https://doi.org/10.23925/2237-9657.2025.v14i1p120-144

A proposal to explore geometry with GeoGebra: graphical exploration and formal demonstration of the collinearity of the barycenters of a polygon

Uma proposta para explorar a geometria com o GeoGebra: exploração gráfica e demonstração formal da colinearidade dos baricentros de um polígono

SAULO MOSQUERA LÓPEZ[1]
https://orcid.org/0000-0002-5932-5446

MARLIO PAREDES[2]
https://orcid.org/0000-0002-9375-3743

WALTER CASTRO[3]
https://orcid.org/0000-0002-7890-681X

ABSTRACT

This paper illustrates an example of a mathematical activity that teachers and students can replicate to create an experience that resembles professional mathematical activity. We extend the property "Consider a triangle ABC, any straight line   and let A', B', C' be the reflections of the points A, B, C on the straight line   then the barycenters of the triangles ABC, A'BC, AB'C and ABC' are collinear and the line of collinearity is perpendicular to the straight line" for any quadrilateral. It is proved that there are four additional triangles, for a total of eight, whose barycenters are collinear and that there are eleven additional quadrilaterals. Some of the geometric and numerical experiments in which GeoGebra software was used, necessary to extend and demonstrate analogous results for the case of a pentagon, are described, and the corresponding results for an n-sided polygon are generalized and demonstrated.

Keywords: *Classroom activity; barycenter; collinearity.*

RESUMO

Este artigo apresenta um exemplo de atividade matemática que professores e alunos podem replicar para gerar uma experiência que se assemelhe à atividade matemática profissional. Estendemos a propriedade: "Considere um triângulo ABC, qualquer linha reta e deixe A', B', C' serem as reflexões dos pontos A, B, C na linha reta, então os baricentros dos triângulos ABC, A'BC, AB'C e ABC' são colineares e a linha de colinearidade é perpendicular à linha reta" e estendemos esse resultado a qualquer quadrilátero. Foi demonstrado que há quatro triângulos adicionais, em um total de oito, cujos baricentros são colineares e que há onze quadriláteros adicionais. Alguns dos experimentos geométricos e numéricos usando o software GeoGebra necessários para estender e demonstrar resultados análogos para o caso de um pentágono são descritos, e os resultados correspondentes para um polígono de n lados são generalizados e demonstrados.

Palavras-chave: *Atividade em sala de aula; baricentro; colinearidade.*

[1] GESCAS Research Group, University of Nariño, Pasto, Colombia. E-mail: samolo@udenar.edu.co
[2] Dr. Martin Luther King, Jr. Early College, Denver CO, USA. The University of Texas Rio Grande Valley, Edinburg, TX, USA. E-mail: marlio_paredes1@dpsk12.net
[3] University of Antioquia, Medellín, Colombia. E-mail: walter.castro@udea.edu.co

RESUMEN

Este articulo presenta un ejemplo de actividad matemática que profesores y estudiantes pueden replicar para generar una experiencia que asemeja la actividad matemática profesional. Extendemos la propiedad: "Considere un triángulo ABC, cualquier línea recta y sea A', B', C' las reflexiones de los puntos A, B, C sobre la línea recta entonces los baricentros de los triángulos ABC, A'BC, AB'C y ABC' son colineales y la línea de colinearidad es perpendicular a la línea recta" y han ampliado este resultado para cualquier cuadrilátero. Se demuestra que hay cuatro triángulos adicionales, para un total de ocho, cuyos baricentros son colineales y que hay once cuadriláteros adicionales. Se describen algunos de los experimentos geométricos y numéricos en los que se utilizó el software GeoGebra, necesarios para ampliar y demostrar resultados análogos para el caso de un pentágono, y se generalizan y demuestran los resultados correspondientes para un polígono de n lados.

Palabras clave: *Actividad en el aula; baricentro; colinealidad.*

Introduction

Paul Halmos stated in the epilogue of his article, "I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom [...] we will train our students to be better problem-posers and problem-solvers than we are" (Halmos, 1980, p. 524). Problem solving is at the heart of mathematical activity. For Tao (2006),

> Mathematics is a multifaceted subject, and our experience and appreciation of it changes with time and experience. As a primary school student, I was drawn to mathematics by the abstract beauty of formal manipulation and the remarkable ability to repeatedly use simple rules to achieve non-trivial answers. (p. 1)

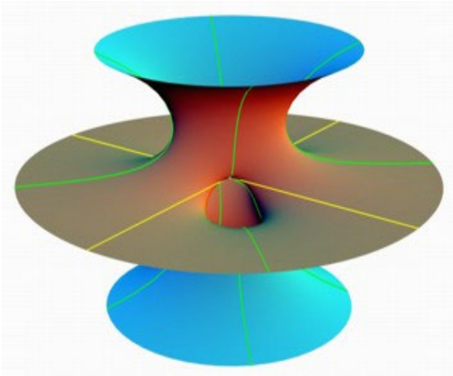
The power of mathematics lies in its relationship between abstraction and concreteness through problem-solving strategies that ultimately connect to the technical problems of society. Technological advances are based on diverse forms of mathematical nature. This power of mathematics has promoted the ubiquitous teaching of mathematics in schools, starting in the early grades. Brown &Walter (1983) propose that teachers initiate students in mathematics education by posing problem-posing activities.

Several mathematical researchers (Goldenberg et al., 1998; Smid, 2022) consider that mathematics teaching should promote student mathematical activity by replicating professional mathematicians' mathematical activity. Several studies (Cuoco et al., 1996; Goldenberg et al., 2003) report on the "habits of mind" that students develop through the study of mathematics and that can be used in the solution of everyday problems.

This proposal requires considering the epistemic nature of mathematics in terms of 'knowledge in action'; knowledge helps to solve problems, thus moving from declarative knowledge to problem-solving ability.

A proposal in mathematics education suggests using mathematical real problems to reproduce school conditions to promote students' mathematical education (Anhalt & Cortez, 2015; Blum & Borromeo, 2009; English, 2009). Designing actual mathematical activities can reproduce conditions for students to explore school mathematical knowledge, solutions and conjectures on novel problems. The exploration of school and non-school mathematical problems is aided by technology in the form of computers and software (Mathematica, GeoGebra, Cabry Geometre). The properties of Costa's surface (Costa, 1984) Figure 1, described analytically, are not easily understandable. Still, their geometrical representation allows us to understand them.

Figure 1: Costa surface



Source: *https://minimal.sitehost.iu.edu/maze/costa.html*

For example, the Costa surface is a complete minimal embedded surface of finite topology (i.e., it has no boundary and does not intersect itself) discovered in 1982 by the Brazilian mathematician Celso José da Costa (Costa, 1982, 1984). It has genus 1 with three punctures (Schwalbe & Wagon, 1999). The fact that it is a surface of topology finite, means that it can be formed by puncturing a compact surface. Topologically, it is a thrice-punctured torus.

The minimal surface discovered in his doctoral thesis solved an old problem in the area of minimal surfaces. Specifically, he found a fourth minimal surface, now known internationally as Costa's Surface. A surface immersed in Euclidean space $\mathbb{R}^3$ is said to be minimal if every point on the surface has a neighborhood that is a surface of smaller area concerning its edge. In this sense, such surfaces are the two-dimensional generalization of geodesics (Academia Brasileira de Ciências [ABC], 2000; Weisstein, 2024). The three minimal surfaces, previously known, were the plane, the catenoid (Euler-1764), and the helicoid (Meusnier-1776).

The Costa surface evolves from a torus, which is deformed until the planar end becomes catenoidal. Defining these surfaces on rectangular tori of arbitrary dimensions yields the Costa surface. Its discovery triggered research and discovery into several new surfaces and opened new conjectures in topology.

This famous surface was described by Costa in his doctoral dissertation (Costa, 1982). He constructed an example of a complete minimal immersion of the torus punctured at three points in $\mathbb{R}^3$ with embedded ends, also proved that the total curvature of such an immersion is -12π (Costa, 1984). Costa described geometrically the surface but didn't prove that it is embedded in $\mathbb{R}^3$, the proof of this fact was given by Hoffman &Meeks (1990). We would like to cite the following paragraph taken from Hoffman & Meeks (1985):

> Costa proved that the surface was complete, and it followed from the results of Jorge & Meeks (1983) that the surface was embedded outside of a compact ball. Our work was motivated by this example. In particular, *we used a computer to explicitly determine the coordinate functions of this surface and draw it*[1]. From the pictures we could see that the surface was embedded, possessed dihedral symmetry, and contained two orthogonal lines. We then found mathematical proofs of these observations.

It is clear how important it is to use a computer to demonstrate a significant mathematical result.

Quarteroni (2022) gives several examples of problems - weather forecast models and mathematics of contagion - that have been modeled with mathematics. The intricate patterns of fractals (Novak, 2004) would not show their beauty to the general public were it not for the help of computers. For Salsa (2016), constructing a mathematical model is based on two main ingredients: general laws and constitutive relations. Identifying constitutive relations may be the most challenging part of investigative mathematical work, but using the computer can simplify the task.

This article shows the case of a Euclidean geometric conjecture and the subsequent relationship between the graphical and numerical exploration, using GeoGebra software, and the formal demonstration of the conjectured patterns. The research links to the trends identified by Gökçe & Güner (2022) on using GeoGebra to improve students' mathematics learning, the analytics of teaching- the role of GeoGebra in mathematics teaching-, and the technological approach -the technological and procedural issues-. The paper shows the close relationship between posing conjectures, their study, and the visual aid offered by GeoGebra software (Pujiyanto & Lo, 2024). One of the main characteristics of a dynamic geometry software-DGS- is interactivity, it must allow the construction, manipulation, exploration, analysis, visualization and transformation of geometric objects in the plane and in space, which enables experimentation and verification of the existing mathematical relationships between geometric objects and therefore the validation of conjectures. These properties are present in the main SGD such as Cabri, the Geometer's Sketchpad and GeoGebra, however, the latter has additional features that distinguish it from the others: It is a free software that is based on java which makes it a multiplatform software that works on various operating systems such as Windows, Linux, Mac, Ubuntu and additionally on Android devices. It has a dual window that allows establishing relationships between the algebraic and graphical representation of an object through an interface consisting of an algebraic window and a graphical window, between which there is a correspondence, since an object defined algebraically has a representation in the graphical window and reciprocally, this means that any modification of an object in one of them is immediately reflected in the other, and finally, GeoGebra has an alternative that allows creating an applet with the construction so that it can be uploaded to the internet and worked from there without the need to have the software installed, in addition the software can be run online or downloaded and used offline.

The explored results, conjectures, and demonstrations are not reported in the mathematical literature[4] as known results. Even for well-known domains, it is possible to discover and prove new results with the graphical help of the software. These types of problems can be replicated as activities in mathematics classes. This proposed use of GeoGebra requires knowledge of the use of GeoGebra, as well as a computer lab where students can explore the illustrations provided by the teacher, act on them, and validate the teacher's assertions. However, it has the value of illustrating possible strategies for both teachers and students, as well as informing the possibility of discovering geometric properties not reported in the literature, whose exploration and demonstration can be done via software that offers a representation system that encourages students to appreciate the use of properties, validate the link between symbolic representations with the semantic content associated with them.

The exploration and demonstration of mathematical results, new or known, is an opportunity for teachers and students to provide mathematical training opportunities and replicate the mathematical activity of professional mathematicians in the classroom. The following shows a new theorem in Euclidean geometry, the statement of a conjecture, and its subsequent exploration and proof using GeoGebra, which can be replicated in the geometry classroom. Suggestions are then offered for using it

[4] Members of the mathematical community were consulted about the result, which they said was unknown and, at the very least, novel.

as a mathematical discussion exercise. The GeoGebra archives are in https://www.geogebra.org/m/ndqxkmnb, for the teachers to explore, modify and use them with the students.

If we accept that a "mathematical object" is defined by the concurrence of situations, techniques, representations, concepts, propositions, and theoretical arguments (Godino et al., 2007), then the configuration of mathematical ideas is complex in which any help oriented to the construction of a semantic network of related "meanings" is welcome. Visualization describes producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand-drawn or computer-generated (Zimmermann & Cunningham, 1991).

Representing mathematical objects is an essential didactic resource to help students solve mathematical problems. For some authors, dynamic exploration is key in teaching mathematics (Boero et al., 1996; Simon, 1996). Visualization and dynamic exploration associated with problems favor producing and testing conjectures and help in selecting "fields of experience" and tasks where such dynamic exploration is "natural" for students (Boero, 1999). The availability of computational resources allows for representing, changing representations, validating conjectures, and proposing a demonstration. Several authors consider demonstration important in students' mathematical experiences (Stylianides & Stylianides, 2006; Yackel & Hanna, 2003).

Appropriate mediation by the teacher is called for in organizing the sequence of graphics, illustrations, and the structure of mathematical proofs as texts, producing and proving conjectures. For Boero (1999), it is required to consider the continuity between the production of a conjecture and the construction of its demonstration (Garuti et al., 1996). By developing exploration, visualizing, inferring, establishing, and eventually demonstrating, students can perform actions proper to mathematical activity, allowing them to understand their assertions and findings (Balacheff, 2002; Stylianides, 2007).

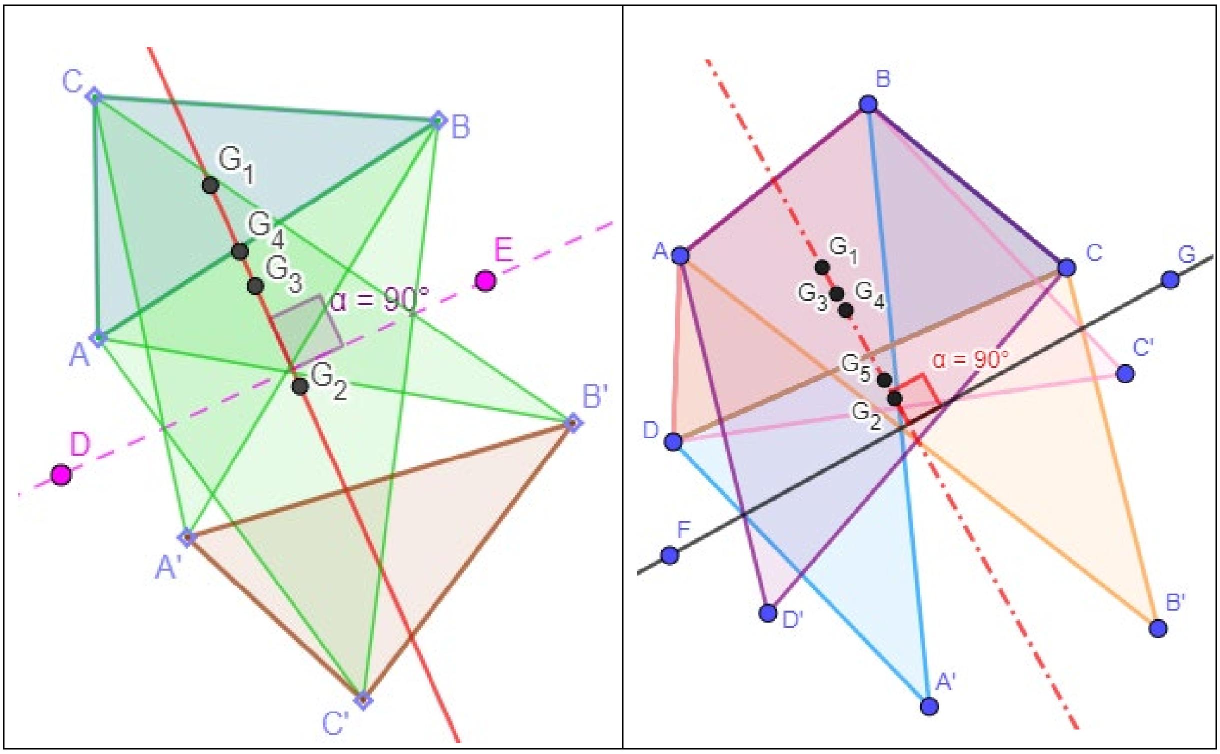
Bonelo-Ayala et al. (2024), using GeoGebra, discussed in his paper the following results.

1. Let's consider a triangle ABC, an arbitrary line $\overleftrightarrow{ED}$ and be A', B', C' the reflections of points A, B, C over the line $\overleftrightarrow{ED}$, then the barycenters of the triangles $ABC, A'BC, AB'C$ and ABC' are collinear and the collinearity line is perpendicular to the line $\overleftrightarrow{ED}$.
2. Let's consider a quadrilateral $ABCD$, an arbitrary line $\overleftrightarrow{EF}$ and be A', B', C', D' the reflections of points A, B, C, D over the line $\overleftrightarrow{EF}$, then the barycenters of the quadrilaterals $ABCD, A'BCD, AB'CD, ABC'D$ and $ABCD'$ are collinear and the collinearity line is perpendicular to the line $\overleftrightarrow{EF}$.

Figure 2 illustrates these results. The authors of this paper formulated the question: What happens if instead of considering the orthogonal reflection of each vertex, we consider the triangles with the reflection of two vertices or with the reflection of their three vertices?

For the quadrilateral case, what does happen if all quadrilaterals formed by the combinations of two, three and four reflections of their vertices are considered? Is it possible to extend the results to a pentagon? Or to any polygon?

Figure 2: Collinearity of the 4 barycenters for the triangle and the 5 barycenters for the quadrilateral



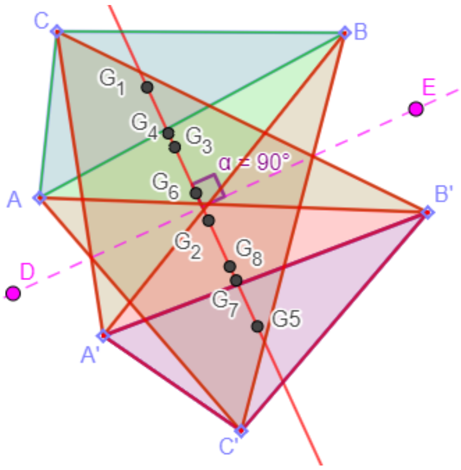
Source: research collection

The second objective of this paper is to formalize the results obtained through the corresponding definitions and demonstrations.

1. The complement of the initial conjecture

For the triangle case, if instead of considering the triangles with the reflection of one vertex, we consider the triangles with the reflection of two vertices, that is, the triangles A'B'C, A'BC', AB'C' and the triangle A'B'C', of the three reflections of its vertices, it is found that the barycenters of these triangles are also collinear on the same collinearity line as those evidenced in the initial conjecture. Therefore, in the triangle case, there are eight collinear barycenters. Figure 3 shows this statement.

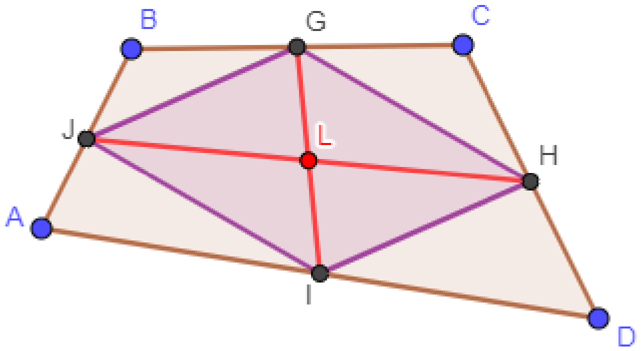
Figure 3: The 8 collinear barycenters for the triangle case



Source: research collection

In order to work with the quadrilateral, it is necessary to mention that Bonelo-Ayala et al. (2024) called the barycenter of a quadrilateral the point of intersection of the quadrilateral diagonals formed by the midpoints of the quadrilateral ABCD.

Figure 4: L is the barycenter of the quadrilateral ABCD



Source: research collection

From Varignon's theorem (Coxeter, 1971) we know the quadrilateral GHIJ is a parallelogram, and its diagonals cross at their middle point L (Figure 4).

If we consider the quadrilateral ABCD and the quadrilaterals obtained by reflecting two of its vertices, this means the quadrilaterals A'B'CD, A'BC'D, A'BCD', AB'C'D, AB'CD', ABC'D'. Also, the quadrilaterals that are obtained by reflecting three of its vertices, this means the quadrilaterals A'B'C'D, A'B'CD', A'BC'D', AB'C'D'. And the quadrilateral given by the reflection of the four vertices, that is A'B'C'D'. Then we get that the initial five barycenters considered and the last eleven barycenters obtained are collinear, for a total of 16 collinear barycenters, see Figure 5.

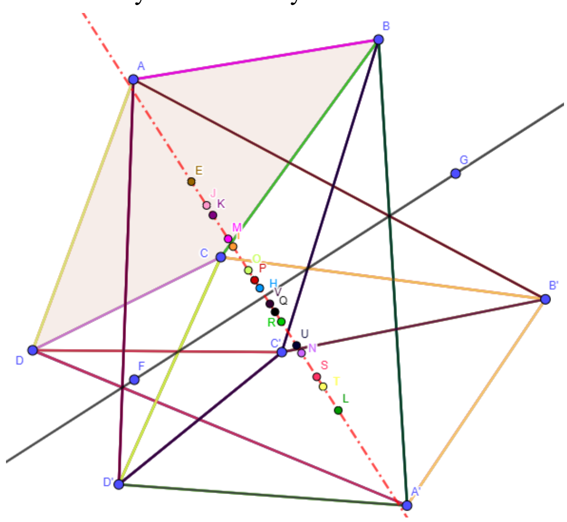
Notice that for the triangle, the number of collinear barycenters can be expressed as

$$1 + 3 + 3 + 1 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3,$$

and for the quadrilateral we have that the number of collinear barycenters can be expressed as

$$1 + 4 + 6 + 4 + 1 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4$$

Figure 5: The collinearity of the 16 barycenters associated to the quadrilateral



Source: research collection

This suggests the possibility to have a conjecture for the number of collinear barycenters of a polygon with n sides if it is possible to define the barycenters for any n-sided polygon.

Conjecture 1. The number of collinear barycenters of a polygon with n sides is 2^n.

For the triangle case, the demonstration of the barycenters collinearity Bonelo-Ayala et al. (2024) used two times the Thales theorem and a property that establishes the barycenter of a triangle divides the median in a certain ratio (Altshiller, 1952). However, this proof can be simplified if we consider that the triangle $A'B'C'$ is the image of the triangle ABC under reflection about the line $\overleftrightarrow{ED}$. Let's see how it works.

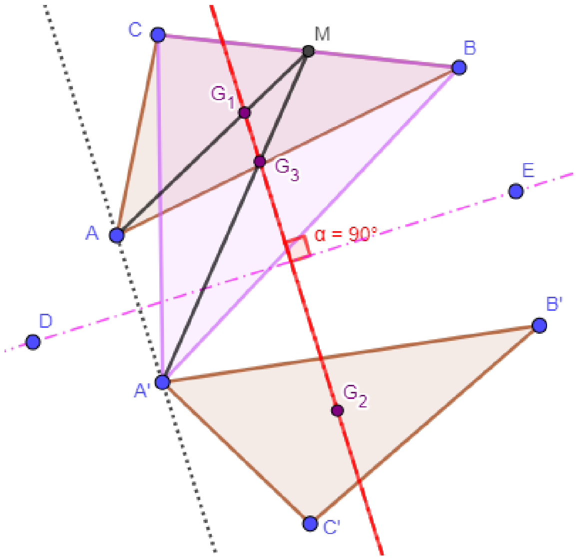
Proposition 1. Let's consider a triangle ABC, an arbitrary line ED, and A', B', C' the reflection of the points A, B, C about the line $\overleftrightarrow{ED}$. Then the barycenters of the triangles $ABC, A'BC, AB'C, ABC', A'B'C, AB'C', A'BC', A'B'C'$ are collinear and the collinearity line is perpendicular to the line $\overleftrightarrow{ED}$. Figure 6 illustrates this result.

Demonstration. We prove the result for triangles given by two of the initial vertices and the reflection of the vertex, the other cases are proved using a similar reasoning. First of all, since a reflection about a line is an isometry then the image of G_1, the barycenter of the triangle ABC, is the point G_2, which is the barycenter of the triangle $A'B'C'$, and by the definition of a reflection about a line, the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{G_1G_2}$ are perpendicular to the line $\overleftrightarrow{ED}$. Now we must prove that the point G_3, the barycenter of the triangle $A'BC$, is on the line $\overleftrightarrow{G_1G_2}$. In order to do this, it is enough to show that the line $\overleftrightarrow{G_1G_3}$ is perpendicular to $\overleftrightarrow{ED}$. Let's call M the middle point of the segment $\overline{BC}$, given that G_1 and G_2 are the barycenter of the triangles ABC and $A'BC$ respectively then

$$\frac{MG_1}{MA} = \frac{1}{3} = \frac{MG_3}{MA'}.$$

By the converse of the Thales' theorem the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{G_1G_3}$ are parallel, therefore the line $\overleftrightarrow{G_1G_3}$ is perpendicular to the line $\overleftrightarrow{ED}$ and by the uniqueness of the perpendicular through a point outside a straight line, the lines $\overleftrightarrow{G_1G_2}$ and $\overleftrightarrow{G_1G_3}$ coincide.

Figure 6: Illustration for the collinearity demonstration in the triangle case



Source: research collection

We will use a similar argument to study the pentagon case and the polygon with n sides case.

2. The pentagon case

This section is divided into two subsections in the first one, some of the numerical and geometrical experiments that allowed defining and consolidating the patterns and characteristics necessary for the case of the pentagon are described. In the second one, these observations are formalized with definitions and with proof of their results.

The numerical and geometrical experiments

We present in three subsections some of the algebraic and geometrical simulations that allowed us to justify the definition of the barycenter of a pentagon, the collinearity of its barycenters, and the metric relations to deal with the pentagon case.

The barycenter

To deal with the polygon case, unlike a triangle or quadrilateral, it is required, first of all, to define its barycenter. After reviewing some geometry texts such as Altshiller (1952) and Martin (1997), no reference for this concept was found; we must adapt known concepts or propose an alternative definition. A first approximation to this concept is obtained from the statistical concept of mean, which is defined as follows (Wayne, 1988).

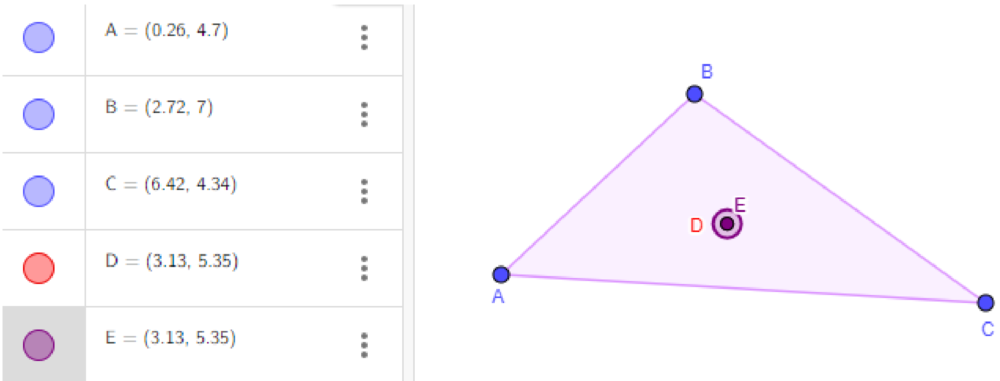
Definition 1. Let X to be a random variable from which n measurements $x_1, x_2, \cdots, x_n$ have been taken, the mean of these measurements is denoted and defined as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Now, the following question arises: what is the relationship between this concept and, for instance, the barycenter of a triangle or the barycenter of a quadrilateral?

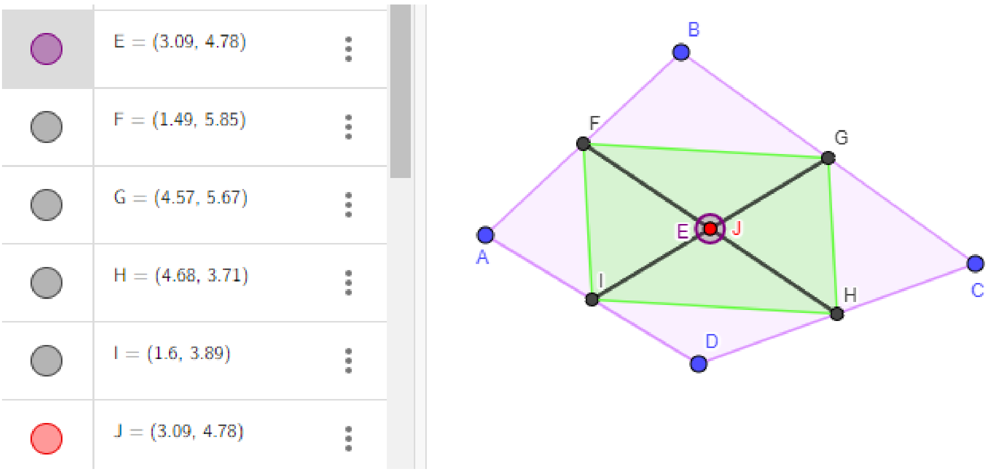
If we use dynamic geometry software like GeoGebra and calculate the corresponding points to the mean of the vertices coordinates of the triangle or the quadrilateral, we can verify that the corresponding points to the barycenter and mean coincide. Figures 7 and 8 illustrate this, it is a simple exercise to demonstrate this result.

Figure 7: The barycenter D and the point E, defined by the mean of coordinates of A, B and C coincide



Source: research collection

Figure 8: The barycenter J and the point E, defined by the mean of coordinates of A, B, C and D coincide



Source: research collection

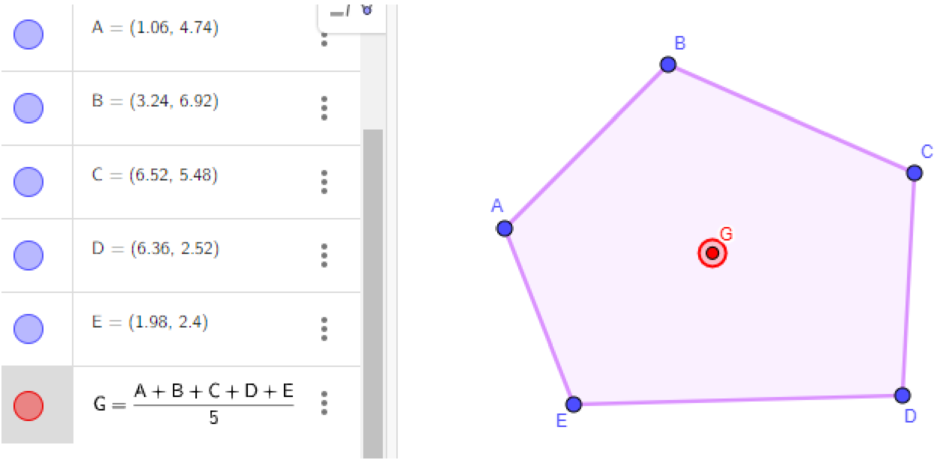
As a consequence of this fact, we can define the barycenter of a pentagon as follows:

Definition 2. Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4), E(x_5, y_5)$ be the coordinates of the vertices of a pentagon, we call the **barycenter** of the pentagon the point G whose coordinates are defined as

$$\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{n}, \frac{y_1 + y_2 + y_3 + y_4 + y_5}{n}\right).$$

Figure 9 illustrates this definition.

Figure 9: The barycenter G of a pentagon



Source: research collection

The collinearity of the barycenters

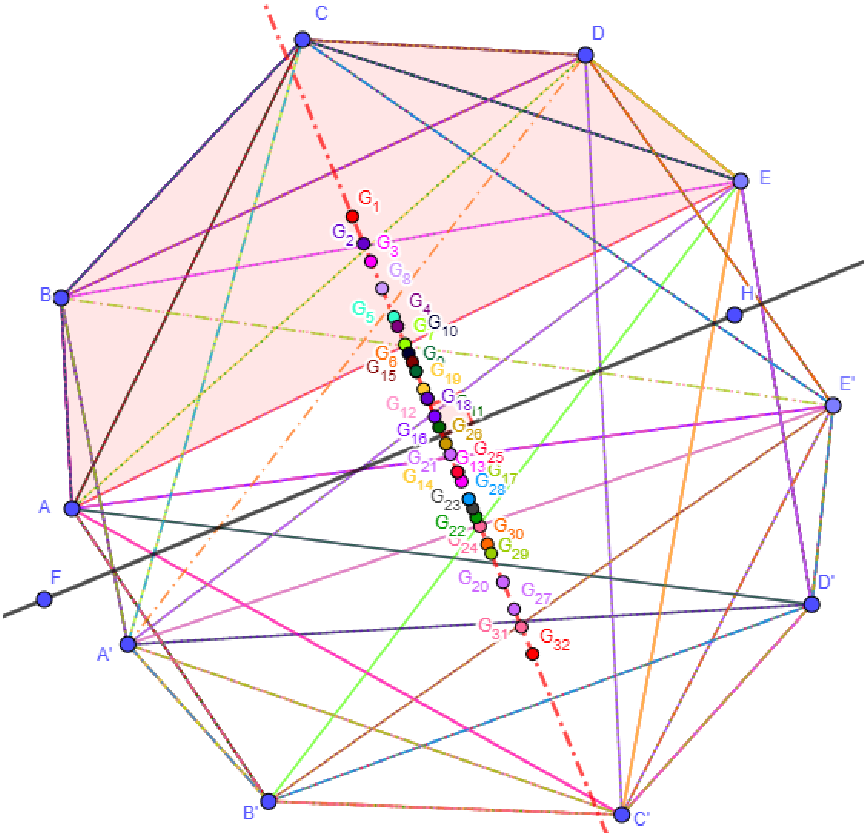
Now with the barycenter definition for a pentagon we use GeoGebra to perform some simulations similarly to the case of the triangle and the quadrilateral. We reflect the vertices of the pentagon on any straight line considering all possible pentagons, their barycenters and, if we expect analogous results to those obtained previously, we must obtain 32 collinear points on a straight line perpendicular to the reflection line.

Conjecture 2. Let's consider a pentagon $ABCDE$, any straight line $l = \overleftrightarrow{FH}$ and A', B', C', D', E' the reflections of the vertices $A, B, C, D,$ and E about the line l, then the thirty-two barycenters $G_1, G_2, G_3, \ldots, G_{31}, G_{32}$ of the pentagons:

$ABCDE, A'B'C'D'E', A'BCDE, AB'CDE, ABC'DE, ABCD'E, ABCDE', A'B'CDE, A'BC'DE,$
$A'BCD'E,\ A'BCDE', AB'C'DE, AB'CD'E, AB'CDE', ABC'D'E, ABC'DE', ABCD'E',$
$A'B'C'DE,\ A'B'CD'E, A'B'CDE', AB'C'D'E, AB'C'DE', ABC'D'E', A'BC'D'E, A'BC'DE',$
$A'BC'DE', A'BC'DE', A'B'C'D'E, A'B'C'DE', AB'C'D'E', A'BC'D'E'$ and $A'B'CD'E'$ are collinear and the line containing those points is perpendicular to the line $l = \overleftrightarrow{FH}$.

To explore this conjecture, we build a tool called Bari-Penta which has as input the vertices of the pentagon and as output the pentagon and its barycenter. We used this tool to simulate with GeoGebra the collinearity. The result is shown in Figure 10, which supports the veracity of the conjecture.

Figure 10: The collinearity of the 32 barycenters related to the pentagon



Source: research collection

The metric relations

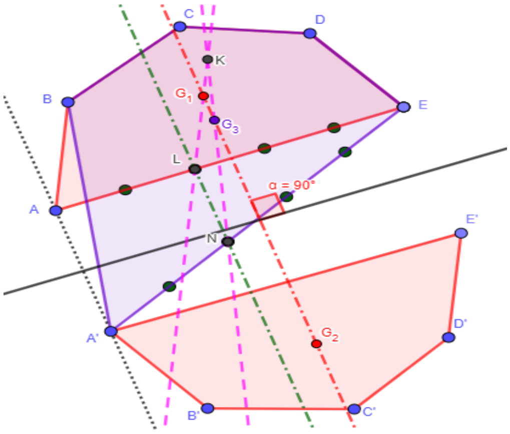
We will formally prove this, for which we consider the converse of the Thales' theorem and some proper metric relations. In the triangle case we have this true statement: *"The three medians of a triangle meet in a point, and each median is trisected by this point"* (Altshiller, 1952), and for the quadrilateral case we have "The diagonals of a parallelogram intersect at their midpoints" (Isaacs, 2001). For the pentagon we don't know a similar result.

Let's describe some of the numerical experiments performed to obtain a useful relation to write a formal demonstration.

Initially we consider a pentagon $ABCDE$, its barycenter G_1, $l = \overleftrightarrow{FH}$ any straight line, A', B', C', D', E' the reflections of A, B, C, D, E about the line l, the pentagons $A'B'C'D'E'$, $A'BCDE$ and G_2, G_3 their respective barycenters. Since a reflection about a line is an orthogonal transformation, it follows that the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{G_1G_2}$ are perpendicular to the line $\overleftrightarrow{FH}$. Now we look for an algebraic relation that guarantees us the point G_3 is on the line $\overleftrightarrow{G_1G_2}$.

a. First, we divide the segments $\overline{AE}$ and $\overline{A'E}$ in five equal parts, we consider for example the points L and N located at two fifths from the vertices A and A' and the line $\overleftrightarrow{LN}$. The construction guarantees that the lines $\overleftrightarrow{AA'}$, and $\overleftrightarrow{LN}$ are parallel. For that reason, we look for relationships ensuring the parallelism of the lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{G_1G_2}$, for which we consider the lines $\overleftrightarrow{LG_1}$, $\overleftrightarrow{NG_3}$ and their intersection point K, see Figure 11.

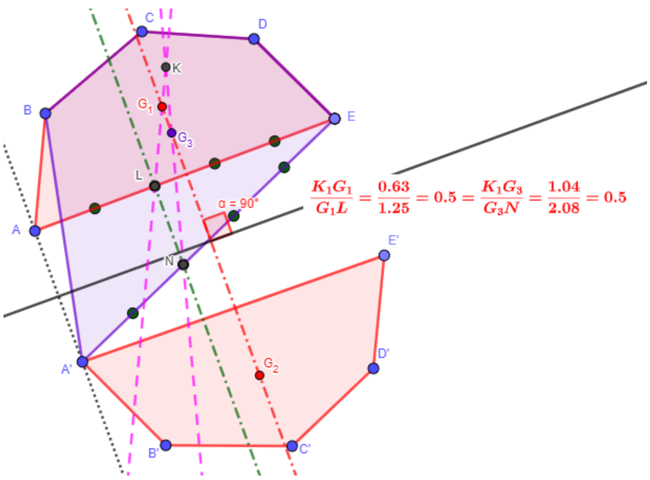
Figure 11: Initial approach looking for metric relationships



Source: research collection

b. Now we consider the segments KG_1, G_1 L, KG_3, G_3 N, the ratios (KG_1)/ (G_1 L), (KG_3)/ (G_3 N) and using GeoGebra we found out these ratios are constant and equal as illustrated in Figure 12 and the dragging test confirms this fact.

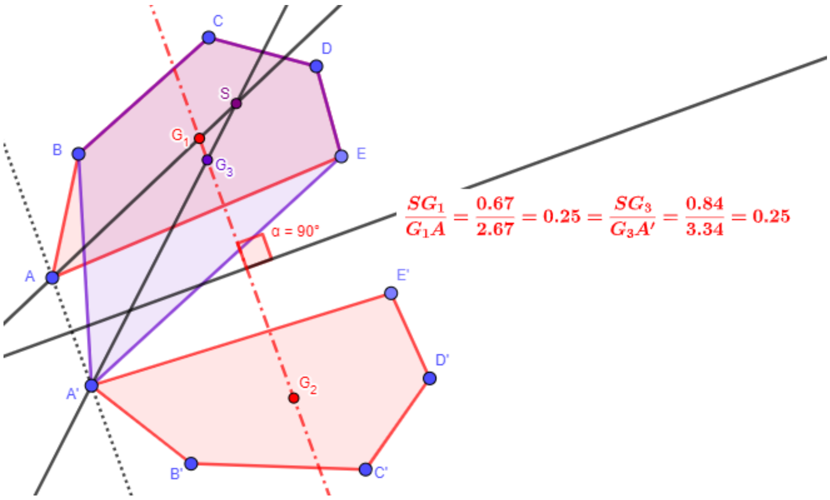
Figure 12: The ratio between the considered segments is constant



Source: research collection

c. Before trying to obtain a formal proof, we would like to respond to the question: What is the geometric meaning of point K? Is it the midpoint of some particular segment? Is it the barycenter of any particular triangle or quadrilateral? Using GeoGebra, some midpoints and some barycenters of triangles and quadrilaterals are calculated, but the results are not conclusive, and no property is identified.
d. Analogous results are obtained if the points located one fifth, three fifths and four fifths from points A and A' are considered, so new alternatives are sought.
e. We consider now the straight lines $\overleftrightarrow{AG_1}$, $\overleftrightarrow{A'G_3}$, the intersection point S, the segments KG_1, G_1A, SG_3, G_3A' and the ratios $\frac{SG_1}{G_1A}, \frac{KS}{G_3A'}$, and a result similar to the previous one is obtained, which is illustrated in Figure 13.

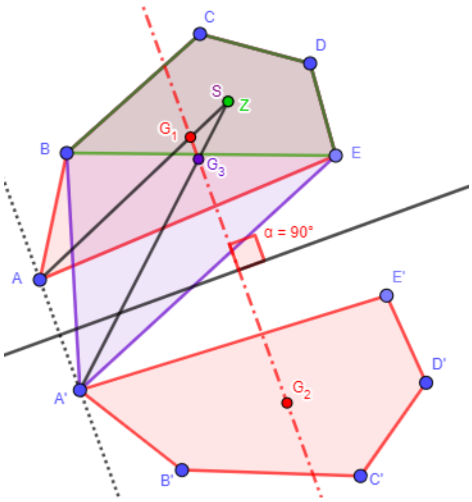
Figure 13: New attempt looking for metric relationships



Source: research collection

f. It seems we are in a similar situation than item c. What is the geometric meaning of point S? However, in this case we are lucky, we calculated the barycenter Z of the quadrilateral BCDE and surprisingly $S = Z$. Figure 14 illustrates this situation.

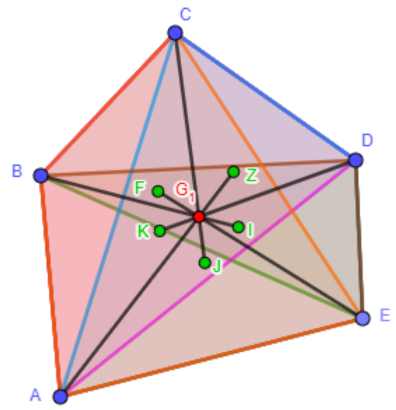
Figure 14: The point S, intersection of the lines $\overleftrightarrow{AG_1}$, $\overleftrightarrow{A'G_3}$, and the barycenter Z of the quadrilateral $BCDE$



Source: research collection

g. Based on the graphical confirmation, we begin with the task of obtaining a formal proof, but a question arises: What happens if we additionally consider the barycenters of the quadrilaterals $ABCD, ACDE, ABDE, ABCE$, and the segments joining the remaining vertex and the corresponding barycenter of the quadrilateral? The result obtained is illustrated in Figure 15.

Figure 15: The concurrency of the segments joining the vertices and the barycenters

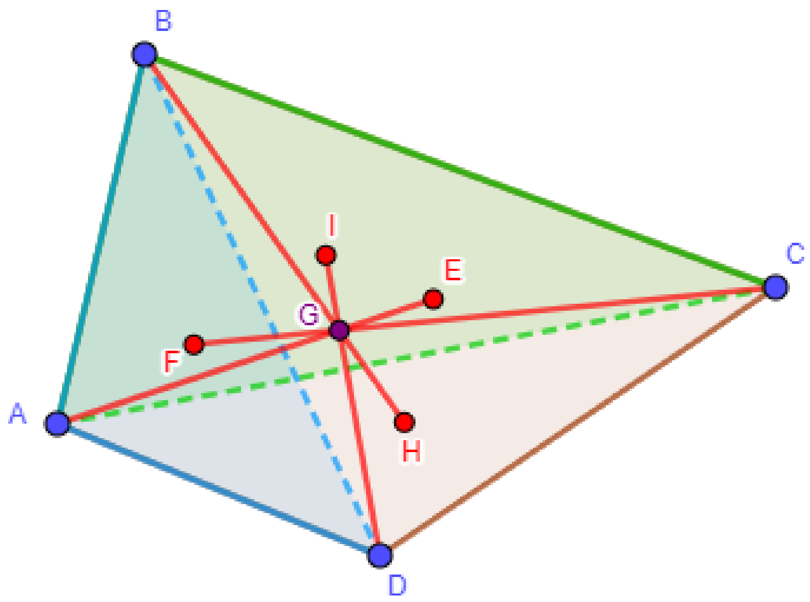


$$\frac{HG_1}{G_1A} = \frac{0.67}{2.67} = 0.25 = \frac{IG_1}{G_1B} = \frac{0.48}{1.92} = 0.25 = \frac{JG_1}{G_1C} = \frac{0.55}{2.18} = 0.25 = \frac{KG_1}{G_1D} = \frac{0.49}{1.96} = 0.25 = \frac{HG_1}{G_1E} = \frac{0.57}{2.27} = 0.25$$

Source: research collection

This allows us to establish a new conjecture for a pentagon with an analogous result for a quadrilateral, which is illustrated in Figure 16.

Figure 16: The concurrency of the segments joining the vertices with the barycenters in a quadrilateral



Source: research collection

Conjecture 3. In a pentagon $ABCDE$ the segments joining the barycenters I, J, K, H, Z of the quadrilaterals $ACDE, ABDE, ABCE, ABCD, BCDE$ with the vertices B, C, D, E, A are concurrent at the barycenter G_1 of the pentagon and satisfy the relation

$$\frac{ZG_1}{G_1A} = \frac{IG_1}{G_1B} = \frac{JG_1}{G_1C} = \frac{KG_1}{G_1D} = \frac{HG_1}{G_1E} = \frac{1}{4}.$$

Formalization of the conjectures for the pentagon

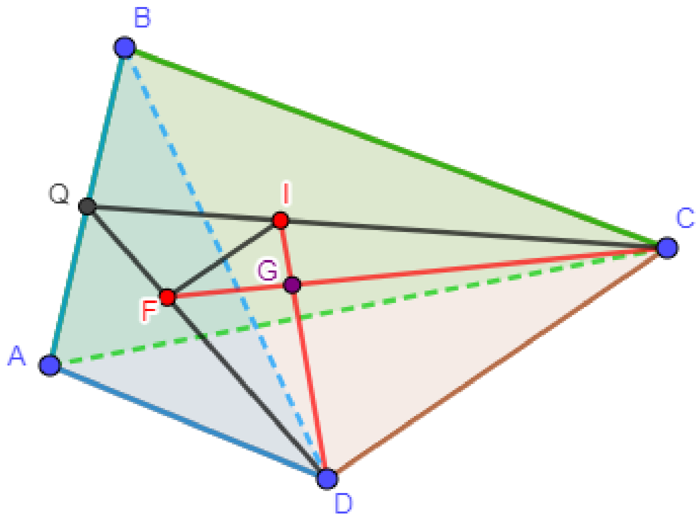
In this section, we present the definitions, statements and proofs that allow us to establish the validity of the conjectures formulated for the pentagon. However, first of all, we formulate and prove a valid result for a quadrilateral, which is necessary for the demonstration of the analogous property for the pentagon.

Proposition 2. Let $ABCD$ be a quadrilateral, E, H, I, F, the barycenters of the triangles BCD, ACD, ABC, ABD, respectively. Then the segments $\overline{AE}, \overline{HB}, \overline{FC}, \overline{ID}$ are concurrent at a point G and the following relation is satisfied

$$\frac{EG}{GA} = \frac{HG}{GB} = \frac{IG}{GD} = \frac{FG}{GC} = \frac{1}{3}.$$

Demonstration. Let Q be the midpoint of the segment $\overline{AB}$ and G the intersection point of the segments $\overline{DI}$ and $\overline{CF}$. Given that $\overline{CQ}$ and $\overline{DQ}$ are medians of the triangles ABC and ABD respectively, see Figure 17, then $\frac{QI}{QC} = \frac{1}{3}$ and $\frac{QF}{QD} = \frac{1}{3}$, this means $\frac{QI}{QC} = \frac{QF}{QD}$.

Figure 17: Illustration of the demonstration



Source: research collection

By the converse of Thales' theorem, we have that $\overline{IF}$ and $\overline{CD}$ are parallel and $\frac{IF}{CD} = \frac{QF}{QD} = \frac{1}{3}$. Since $\overline{IF}$ and $\overline{CD}$ are parallel, it follows the triangles GFI and GDC are similar, then $\frac{IF}{CD} = \frac{IG}{GD} = \frac{FG}{GC} = \frac{1}{3}$.

If we now consider the triangles ABC and DBC, their barycenters I, E and if we denote with T the intersection point of the segments $\overline{DI}$ and $\overline{AE}$ then with an analogous argument we obtain that $\frac{IE}{AD} = \frac{IT}{TD} = \frac{ET}{TA} = \frac{1}{3}$, hence $\frac{IT}{TD} = \frac{1}{3} = \frac{IG}{GD}$, therefore the points T and G coincide.

With a similar argument finally, we obtain.

$$\frac{EG}{GA} = \frac{HG}{GB} = \frac{IG}{GD} = \frac{FG}{GC} = \frac{1}{3},$$

and consequently, the segments $\overline{AE}, \overline{HB}, \overline{FC}, \overline{ID}$ are concurrent at the point G.

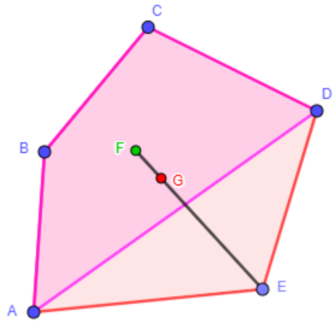
We observed that the previous relation can expressed as

$$\frac{EG}{EA} = \frac{HG}{HB} = \frac{FG}{FC} = \frac{IG}{ID} = \frac{1}{4}.$$

This relation will be used to prove the analogous proposition for a pentagon.

Definition 3. The median of a pentagon is the segment that joins a vertex to the barycenter of the quadrilateral formed by the other four vertices, Figure 18.

Figure 18: The median EF of the pentagon $ABCDE$



Source: research collection

The following result presents a property for the medians of a pentagon.

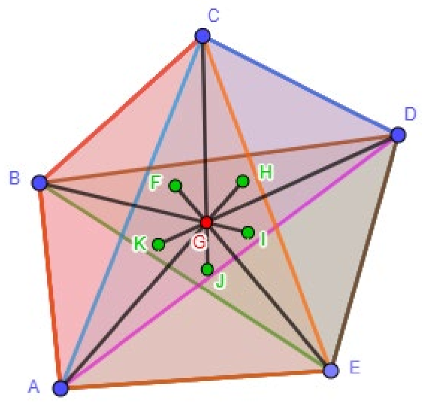
Proposition 3. The medians of a pentagon are concurrent at a point G (Figure 19) and this point divides the medians at the ratio $\frac{1}{4}$ from the barycenter of the corresponding quadrilateral, i.e. if AH, BI, CK, DJ and EF are the medians of the pentagon $ABCDE$ and G is the concurrency point then

$$\frac{HG}{GA} = \frac{IG}{GB} = \frac{JG}{GC} = \frac{KG}{GD} = \frac{FG}{GE} = \frac{1}{4},$$

or equivalently

$$\frac{HG}{HA} = \frac{IG}{IB} = \frac{JG}{JC} = \frac{KG}{KD} = \frac{FG}{FE} = \frac{1}{5}$$

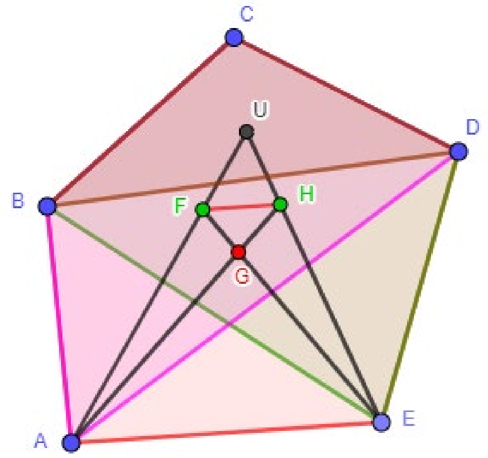
Figure 19: The concurrency of the medians of a pentagon



Source: research collection

Demonstration. Let F and H be the barycenters of the quadrilaterals $ABCD$ and $BCDE$, G the intersection point of the medians EF and AH of the pentagon $ABCDE$ and U the barycenter of the triangle BCD, see Figure 20. Using proposition 2 we have $\frac{UH}{HE} = \frac{1}{3} = \frac{UF}{FA}$ and by the converse of the Tales' theorem $\overleftrightarrow{FH}$ and $\overleftrightarrow{AE}$ are parallel then the triangles FHG and EAG are similar. Using the proposition 2 again we get $\frac{UH}{UE} = \frac{1}{4} = \frac{UF}{UA}$, then $\frac{FH}{EA} = \frac{1}{4}$, it follows $\frac{FG}{GE} = \frac{1}{4} = \frac{HG}{GA}$.

Figure 20: An illustration for the demonstration of the proposition 3



Source: research collection

Let us now consider the quadrilaterals $BCDE$ and $CDEA$, and their barycenters H, I. We call W the intersection point of the medians BI, AH of the pentagon $ABCDE$, and T the barycenter of the triangle CDE. Then with a similar argument to the previous one we obtain that $\frac{HW}{WA} = \frac{1}{4} = \frac{IW}{WB}$ and combining this equality with the previous proportion it turns out that $\frac{HW}{WA} = \frac{1}{4} = \frac{HG}{GA}$ therefore the points W and G coincide, which means the medians AH, BI and EF are concurrent at the point G. In the same way, it is shown that the medians CJ y DK intersect at the point G and that $\frac{JG}{GC} = \frac{1}{4} = \frac{KG}{GD}$. This completes the demonstration.

Remark. The last proposition allows us to write the ratio $\frac{HG}{HA}$ as $\frac{HG}{HA} = \frac{HG}{HG+GA} = \frac{HG}{HG+4HG} = \frac{1}{5}$, so

$$\frac{HG}{HA} = \frac{IG}{IB} = \frac{KG}{KC} = \frac{JG}{JD} = \frac{FG}{FE} = \frac{1}{5},$$

which means the point G is determined by the expression,

$$G = \frac{A + B + C + D + E}{5}.$$

Therefore, if the coordinates of points A, B, C, D and E are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$, and $E(x_5, y_5)$ respectively then the coordinates of the pentagon barycenter G are

$$G\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}, \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5}\right).$$

This justifies the definition 2 in section 3.1.1. Now we can define the barycenter of a pentagon as follows.

Definition 4. We call **barycenter** of a pentagon to the concurrency point of the medians.

The following result is obtained concerning the collinearity of the barycenters of all pentagons, which is obtained by reflecting on a straight line and combining the vertices of a pentagon $ABCDE$..

Theorem 1. Let's consider a pentagon $ABCDE$, any straight line $l = \overleftrightarrow{FH}$ and let A', B', C', D', E' be the reflections of A, B, C, D and E about the line l then the thirty-two barycenters $G_1, G_2, G_3, \ldots, G_{31}, G_{32}$ of the pentagons

$ABCDE, A'B'C'D'E', A'BCDE, AB'CDE, ABC'DE, ABCD'E, ABCDE', A'B'CDE,$

$A'BC'DE, A'BCD'E, A'BCDE', AB'C'DE, AB'CD'E, AB'CDE', ABC'D'E, ABC'DE',$

$ABCD'E', A'B'C'DE, A'B'CD'E, A'B'CDE', AB'C'D'E, AB'C'DE', ABC'D'E', A'BC'D'E,$

$A'BC'DE', A'BC'DE', A'BC'DE', A'B'C'D'E, A'B'C'DE', AB'C'D'E', A'BC'D'E'$ and

$A'B'CD'E'$

are collinear and the line containing those points is perpendicular to the line $l = \overleftrightarrow{FH}$.

Demonstration. The proof will be carried out for pentagons made up of four of the initial vertices and a reflection of the remaining vertex, for this case, we will consider the pentagon $A'BCDE$-Figure 21- the other cases will be considered in section 4 where we will treat the general case.

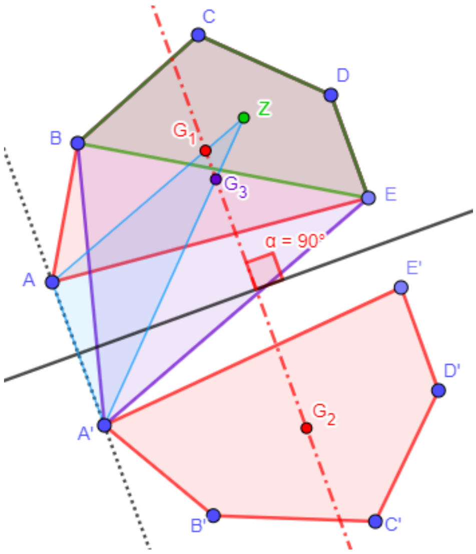
First, we observe that since a reflection about a line is an orthogonal transformation, then the image of point G_1, barycenter of the pentagon $ABCDE$, is the point G_2, barycenter of the pentagon $A'B'C'D'E'$. By the definition of a reflection about a line, the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{G_1G_2}$ are perpendicular to the line $\overleftrightarrow{FH}$. Now we must prove that the point G_3, barycenter of the pentagon $A'BCDE$ is over the line $\overleftrightarrow{G_1G_2}$, and to get this it is sufficient to demonstrate that the line $\overleftrightarrow{G_1G_3}$ is perpendicular to the line $\overleftrightarrow{FH}$.

Let Z be the barycenter of the quadrilateral $BCDE$, then AZ is a median of the pentagon $ABCDE$ and $A'Z$ is a median of the pentagon $A'BCDE$. Therefore,

$$\frac{ZG_1}{G_1A} = \frac{1}{4} \quad \text{and} \quad \frac{ZG_3}{G_3A'} = \frac{1}{4}.$$

So, $\frac{ZG_1}{G_1A} = \frac{ZG_3}{G_3A'}$, and by the converse of the Thales' theorem the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{G_1G_3}$ are parallel. Then $\overleftrightarrow{G_1G_3}$ is perpendicular to the line $\overleftrightarrow{FH}$ and by the uniqueness of the perpendicular through a point exterior to a line we conclude that the lines $\overleftrightarrow{G_1G_2}$ and $\overleftrightarrow{G_1G_3}$ are the same.

Figure 21: An illustration to demonstrate the collinearity of the barycenters of a pentagon



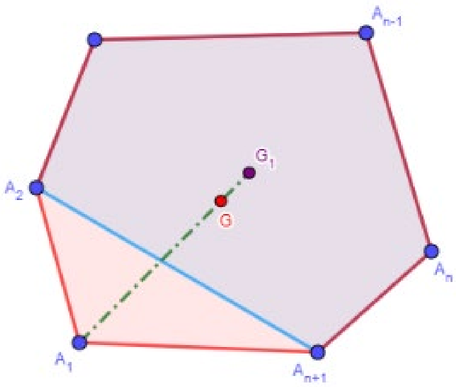
Source: research collection

3. The generalization to a polygon with n sides

In this section, we consider last question the last question asked in the introduction: Is it possible to extend these results to any polygon? The answer is affirmative, and the concepts, propositions and demonstrations are presented below.

Definition 5. It is called median of a polygon with n sides the segment joining the barycenter of the polygon with $n-1$ sides with the remaining vertex, Figure 22.

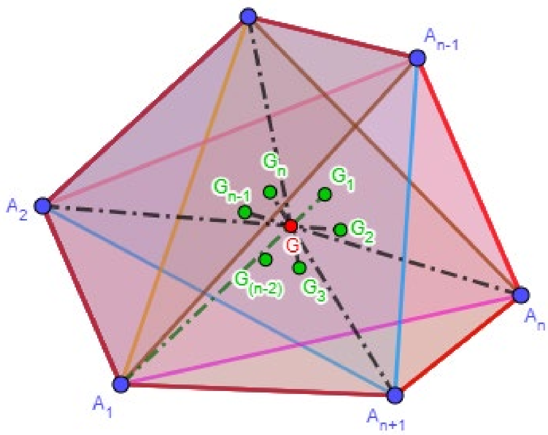
Figure 22: A median of a n-polygon



Source: research collection

It is clear that a polygon with n sides possess n medians, and we have the following result, which is proved using mathematical induction.

Figure 23: The barycenter G of a n-polygon



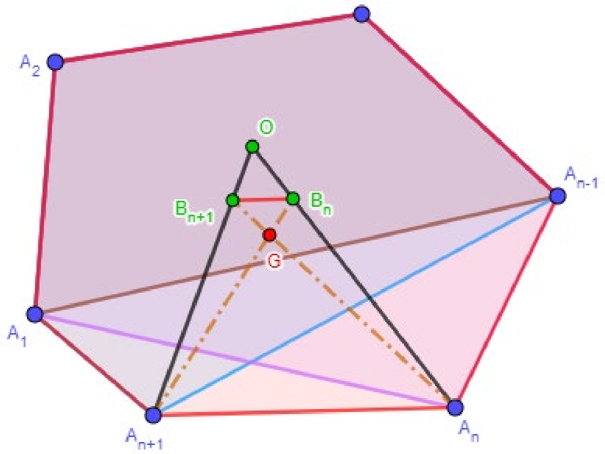
Source: research collection

Theorem 2. The medians of a polygon with n sides are concurrent at a point G which divides each median in the ratio $\frac{1}{n-1}$ from the barycenter of the polygon with $n-1$ sides, Figure 23.

Demonstration. Let's suppose that the theorem is true for polygons with k sides, for $k \leq n$, and we must prove the result for polygons with $n+1$ sides.

Let $A_1, A_2, \dots, A_n, A_{n+1}$ be the vertices of the polygon with $n+1$ sides, and O the concurrency point of the medians of the polygon with $n-1$ sides $A_1A_2 \dots A_{n-1}$. Then the segments $A_{n+1}O$ and A_nO are medians of the polygons with n sides $A_1A_2, \dots A_{n-1}A_n$ and $A_{n+1}A_1A_2, \dots A_{n-1}$.

Figure 24: Illustration for the demonstration of theorem 2



Source: research collection

If B_n, B_{n+1} are the concurrency points for the medians of the polygons with n sides $A_1A_2 \dots A_{n-1}A_n$ and $A_1A_2 \dots A_{n-1}A_{n+1}$, and G is the concurrency point of the medians of the polygon $A_1A_2 \dots A_nA_{n+1}$, in particular it is the intersection point of the medians $A_{n+1}B_n$ and A_nB_{n+1}then by the induction hypothesis we have

$$\frac{OB_{n+1}}{B_{n+1}A_{n+1}} = \frac{1}{n-1} = \frac{OB_n}{B_nA_n}.$$

Using the converse of the Thales' theorem the lines $\overleftrightarrow{B_{n+1}B_n}$ and $\overleftrightarrow{A_{n+1}A_n}$ are parallel, therefore the triangles $OB_{n+1}B_n$ and $OA_{n+1}A_n$ are similar, see Figure 24. From the induction hypothesis follows that $\frac{OB_{n+1}}{OA_{n+1}} = \frac{1}{n} = \frac{OB_n}{OA_n}$, then $\frac{B_{n+1}B}{A_{n+1}A} = \frac{1}{n}$, from this we can conclude that

$$\frac{B_{n+1}G}{GA_n} = \frac{1}{n} = \frac{B_nG}{GA_{n+1}}.$$

Similarly, we can prove the other ratios and the concurrency of the other medians at the point G, which complete the demonstration of our theorem.

This result allows us to generalize the definition of the barycenter of a polygon in the following way.

Definition 6. We call barycenter of the polygon with n sides to the concurrency point G of the medians.

Finally, we establish and prove the main theorem of this work. This result gives us the generalization of the collinearity of the barycenters of a polygon with n sides related to the reflection of the vertices of a polygon about a straight line.

Theorem 3. Let's consider a polygon with n sides $A_1A_2 \dots A_n$, any straight line $\overleftrightarrow{ED}$ and let $A'_1, A'_2, \dots, A'_n$ be the reflections of the vertices of polygon about this line, then the 2^n barycenters of the polygons $A_1A_2, \dots A_n$, $A'_1A_2 \dots A_n$, … ,$A_1A_2 \dots A'_n$, $A'_1A'_2 \dots A_n$, … ,$A_1 \dots A'_{n-1}A'_n$, … …, $A'_1A'_2 \dots A'_n$ are collinear and the collinearity line is perpendicular to the line $\overleftrightarrow{ED}$.

Demonstration. The proof is done in two steps: first, the statement about the number of barycenters and second, their collinearity.

The number of barycenters. The polygon $A_1A_2 \dots A_n$ has a unique barycenter which can be expressed as $1 = \binom{n}{0}$. By considering the polygons made up by $n-1$ vertices and the reflection about the line $\overleftrightarrow{ED}$ of the remaining vertex, this means polygons of the form $A_1'A_2 \dots A_n, \ \dots \ , A_1A_2 \dots A_n'$. The number of polygons with $n-1$ vertices is the number of ways to choose 1 element from a set of n elements, it is $\binom{n}{1}$ and consequently we have $\binom{n}{1}$ barycenters.

Similarly, if we consider the polygons made up by $n-2$ vertices and the reflection about the line $\overleftrightarrow{ED}$ of the two remaining vertices, this means polygons of the form $A_1'A_2' \dots A_n, \ \dots \ , A_1 \dots A_{n-1}'A_n'$, then the number of polygons in this case is the number of ways to choose 2 elements from a set of n elements, it is $\binom{n}{2}$ and consequently the same number of barycenters.

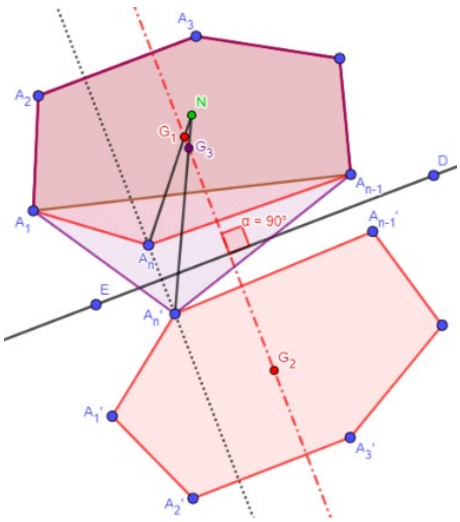
For the polygons made up by $n-k$ vertices and the reflection about the line $\overleftrightarrow{ED}$ of the k remaining vertices we obtain $\binom{n}{k}$ polygons and the same number of barycenters. The polygon $A_1'A_2' \dots A_n'$ has only one barycenter which can be expressed as $1 = \binom{n}{n}$.

Finally, we can conclude that the total number of barycenters is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n.$$

Given that a reflection about a line is an isometry then the image of the point G_1, barycenter of the polygon $A_1A_2 \dots A_n$, is the point G_2, barycenter of the polygon $A_1'A_2' \dots A_n'$. By the definition of a reflection about a line, the straight lines $\overleftrightarrow{A_nA_n'}$ and $\overleftrightarrow{G_1G_2}$ are perpendicular to the line $\overleftrightarrow{ED}$. In order to prove the collinearity of the barycenters we consider several cases according to the number of vertices corresponding to reflections of the original vertices of the polygon, see Figure 25.

Figure 25: Illustration for the demonstration of the collinearity of the barycenters of a n polygon



Source: research collection

Case 1: a vertex of the polygon is the reflection of a vertex of the initial polygon, let's say the polygon $A_1A_2 \dots A_n'$ and we must prove that the point G_3, barycenter of the polygon $A_1A_2 \dots A_n'$ is on the line $\overleftrightarrow{G_1G_2}$. To prove this, it is enough to prove that the line $\overleftrightarrow{G_1G_3}$ is perpendicular to the line $\overleftrightarrow{ED}$.

Let's N be the barycenter of the $n-1$ polygon $A_1A_2 \dots A_{n-1}$, then the segment A_nN is a median of the polygon $A_1A_2 \dots A_n$ and the segment $A_n'N$ is a median of the polygon $A_1A_2 \dots A_n'$. Using the theorem 3, this implies

$$\frac{NG_1}{G_1A_n} = \frac{1}{n-1} \quad \text{and} \quad \frac{NG_3}{G_3A_n} = \frac{1}{n-1}.$$

Then $\frac{NG_1}{G_1A_n} = \frac{NG_3}{G_3A_n}$, and applying the converse of the Thales' theorem we have that the lines $\overleftrightarrow{A_nA'_n}$ and $\overleftrightarrow{G_1G_3}$ are parallel, which implies the line $\overleftrightarrow{G_1G_3}$ is perpendicular to the line $\overleftrightarrow{ED}$. By the uniqueness of the perpendicular through a point exterior to a line we conclude that the lines $\overleftrightarrow{G_1G_2}$ and $\overleftrightarrow{G_1G_3}$ are the same.

Case 2: two vertices of the polygon are the reflection of two vertices of the initial polygon, let's say the polygon $A_1A_2 \dots A'_{n-1}A'_n$ and F_3 its barycenter. We must prove that the point F_3 is on the line $\overleftrightarrow{G_1G_2} = \overleftrightarrow{G_1G_3}$. To prove this, it is enough to consider the polygons $A_1A_2 \dots A'_n$ and $A_1A_2 \dots A'_{n-1}, A'_n$, their corresponding barycenters G_3 and F_3, and repeat the argument used for case 1.

The other cases are treated analogously, considering the polygon with k vertices that are the reflection of k vertices from the initial polygon, the new polygon with $k+1$ vertices reflections of $k+1$ vertices of the initial polygon and their respective barycenters. The latter completes the demonstration.

Conclusions

Dynamic geometry software is a visual experimentation tool that favors identifying patterns and behaviors that promote conjecture (Bonelo-Ayala et al., 2024). Experimentation and dynamic graphic representation precede reasoning, argumentation, and formalization of results. These mathematical activities promote valuable opportunities for mathematics education and can be developed by students with the help of software and teacher management. While formulating a proposal to use GeoGebra to illustrate, explore, and demonstrate geometric properties, this proposal has the advantage of showing the properties geometrically. The teacher can respond to students' concerns and questions by modifying the graphs to validate the students' suggestions. This proposal, however, requires a deep knowledge of the use of GeoGebra and a computer lab where students can modify the graphs provided by the teacher to visualize, explore, conjecture, and demonstrate.

This work shows that with the help of a software, it was possible to represent a conjecture about the barycenters of polygons graphically, and then, based on the graphical evidence, we proceeded to the formal proof of the conjecture. The exploration is centered on known geometric objects, on simple concepts such as parallelism, perpendicularity, triangle, quadrilateral, polygon, reflection about a straight line, barycenter, and similarity, as usual in Euclidean geometry courses. In this way, we see that it is still possible to find novel geometric results not contained in school textbooks, and at the same time provide an example of a mathematical activity that students can replicate and that teachers can guide.

The exploration, the posing of conjectures, the statement of questions, the analysis of graphs, and the search for a formal argumentation show steps of mathematical activity that can be replicated in the classroom. Students can be offered opportunities to visualize, interpret, classify, order, count, measure, appraise, differentiate, abstract, and synthesize, among other actions that professional mathematicians develop in their research. Although the analysis and the results obtained have been developed for convex polygons, the numerical experiments performed suggest that the property of the collinearity of the barycenters, as well as that of the concurrence of the medians and their metric relations, are true for concave and crossed polygons. However, the validity of the demonstrations in these cases should be reviewed. Given the length of this paper, this analysis has not been considered and could be the fruit of future work.

References

Academia Brasileira de Ciências [ABC]. (2000). Celso José da Costa. Academia Brasileira de Ciências. https://www.abc.org.br/membro/celso-jose-da-costa/

Altshiller, N. (1952). *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*. Dover Publications.

Anhalt, C. O., & Cortez, R. (2015). Mathematical modeling: A structured process. *Mathematics Teacher, 108*(6), 446-452. https://doi.org/10.5951/mathteacher.108.6.0446

Balacheff, N. (2002). The researcher epistemology: a deadlock from educational research on proof. In F. L. Lin (Ed.), 2002 International conference on mathematics- understanding proving and proving to understand (pp. 23-44). Taipei, Taiwan: NSC and NTNU

Blum, W., & Borromeo, F. R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*, 45-58.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education, in Preuve. [On line] Available in: http://www.lettredelapreuve.lt/Newsletter/990708Theme/ ES.html.

Boero, P.; Garuti, R. and Mariotti, M.A. (1996). 'Some dynamic mental processes underlying producing and proving conjectures', Proceedings of PME-XX, Valencia, vol. 2, pp. 121-128.

Bonelo-Ayala, Y.; Benítez-Mojica, D., & Muñoz-Posso, J. (2024). Un nuevo teorema geométrico y su aplicación en la construcción de conjeturas a través de un ambiente de geometría dinámica. *Revista Científica,49*(1), 44-61. https://doi.org/10.14483/23448350.20680

Brown, S. I., & Walter, M. I. (1983). *The art of problem posing.* The Franklin Institute Press.

Costa, C. (1982). Imersões minimas completas em R 3 de gênero um e curvatura total finita, Doctoral thesis, IMPA, Rio de Janeiro, Brasil, 1982. (Example of a complete minimal immersion in R 3 of genus one and three ends, Bull. Soc. Bras. Mat., 15 (1984) 47–54.)

Costa, C. (1984). Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. *Boletim da Sociedade Brasileira de Matemática, 15*, 47–54.

Coxeter, H. S. M. (1971). *Fundamentos de geometría*. Editorial Limusa.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of minds: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*, 375-402.

English, L. (2009). Promoting interdisciplinarity through mathematical modelling. *ZDM, 41*, 161-181. https://doi.org/10.1007/s11858-008-0106-z

Garuti, R.; Boero, P.; Lemut, E.& Mariotti, M. A. (1996). 'Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems', Proc. of PME-XX, Valencia, vol. 2, pp. 113-120

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. ZDM – Mathematics Education, 39(1), 127–135. https://doi.org/10.1007/s11858-006-0004-1

Gökçe, S., & Güner, P. (2022). Dynamics of GeoGebra ecosystem in mathematics education. Education and Information Technologies, 27(3), 4181–4198. https://doi.org/10.1007/s10639-021-10836-1 (Retrieved from https://link.springer.com/article/10.1007/s10639-021-10836-1

Goldenberg, E. P., Cuoco, A., & Mark, J. (1998). A role for geometry in general education. In R. Lehrer, & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 3-44). Mahwh, NJ: Lavrance Erlbaum Associates.

Goldenberg, E. P., Shteingold, N., & Feurzeig, N. (2003). Mathematical habits of mind for young children. In F. K. Lester, & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-Grade 6* (pp. 15-29). Reston, VA: National Council of Teachers of Mathematics.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly, 87* (7), 519–524. https://doi.org/10.2307/2321415

Hoffman, D. A., & Meeks III, W. H. (1985). Complete embedded minimal surfaces of finite total curvature. *Bulletin of the American Mathematical Society, 12*(1), 134-136.

Hoffman, D. A., & Meeks III, W. H. (1990). Embedded Minimal Surfaces of Finite Topology. *Annals of Mathematics, 131*(1), 1-34.

Isaacs, M. (2001). *Geometry for college students (The Brooks/Cole Series in Advanced Mathematics).* Brooks/Cole.

Martin, G. E. (1997). *Transformation Geometry – An Introduction to Symmetry.* New York: Springer-Verlag New York Inc.

Novak, M. M. (2004). *Thinking patterns: Fractals and related phenomenon in nature* (1st ed.). World Scientific Publishing.

Pujiyanto, F., & Lo, J. (2024). A synthesis on the integration of computer technology in geometry classes, pp (2054-2061) in Kosko, K. W., Caniglia, J., Courtney, S. A., Zolfaghari, M., & Morris, G. A., (2024). Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Kent State University.

Quarteroni, A. (2022). *Modeling Reality with Mathematics*. Springer.

Salsa, S. (2016). *Partial differential equations in action* (3rd ed.). Springer. https://doi.org/10.1007/978-3-319-31238-5

Simon, M.: 1996, 'Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing', Educational Studies in Mathematics, 30, 197-210

Schwalbe, D., & Wagon, S. (1999). The Costa Surface. In *Mathematica in Education and Research* (Vol. 8, pp. 56-63).

Smid, H. J. (2022). Realistic Mathematics Education. In: *Theory and Practice. International Studies in the History of Mathematics and its Teaching*. Springer. https://doi.org/10.1007/978-3-031-21873-6_8

Stylianides, A. J., & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: The case of reasoning and proving. Proceedings of PME 30, 5, 201–208.

Stylianides, A. J. (2007). Proof and proving in school mathematics. Journal for Research in Mathematics Education, 38, 289–321.

Tao, T. (2006). *Solving mathematical problems: A personal perspective.* Oxford University Press.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), A research companion to principles and standards for school mathematics (pp. 22–44). Reston, VA: National Council of Teachers of Mathematics.

Wayne, D. (1988). *Estadística con aplicaciones a las Ciencias Sociales y a la Educación*. McGraw-Hill.

Weisstein, Eric W. (2024). Costa Minimal Surface. From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/CostaMinimalSurface.html

Zimmermann, W. & Cunningham, S. (1991). What is mathematical visualization? In W. Zimmermann & S. Cunningham, (Eds.), Visualization in teaching and learning mathematics (pp. 1-8). Washington, DC: Mathematical Association of America.

Sent: 03/01/2025

Accepted: 17/04/2025