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Definite integrals of one variable: an intervention proposal with exploratory tasks

Integrales definidas de uno variable: una propuesta de intervención con tareas exploratorias

Intégrales définies d´une variable : une proposition d'intervention avec des tâches exploratoires

Integrais definidas de uma variável: uma proposta de intervenção com tarefas exploratórias

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Abstract

Considering the difficulties that students in the Differential and Integral Calculus discipline have in understanding the concept of a defined integral of a variable, this article aims to investigate the elaboration and implementation of an intervention proposal, based on work with problem solving episodes. tasks, which offers students opportunities to explore this concept. We discuss, as a theoretical foundation, the importance of Riemann integrals and multiplicative base sums in understanding definite integrals. We also provided a characterization of the methodology used in our research, as well as the context of intervention and data collection. The analysis of discussions held in small groups about two exploratory tasks is based on a framework that deals with layers of knowledge, with regard to understanding the concept of defined integrals. As a result, we were able to infer that with the exploratory tasks students were able to substantially explore the concept of multiplicative base sum present in the Riemann sum, in relation to the product, sum and limit layers.

Keywords: Teaching Differential and Integral Calculus, Riemann Integrals, Layers of knowledge, Task solving episodes.

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Resumen

Considerando las dificultades que tienen los estudiantes de la disciplina de Cálculo Diferencial e Integral para comprender el concepto de integral definida de una variable, este artículo tiene como objetivo investigar la elaboración e implementación de una propuesta de intervención, basada en el trabajo con episodios de resolución de tareas, que ofrece a los estudiantes oportunidades para explorar este concepto. Discutimos, como fundamento teórico, la importancia de las integrales de Riemann y las sumas de bases multiplicativas para comprender las integrales definidas. También proporcionamos una caracterización de la metodología utilizada en nuestra investigación, así como el contexto de intervención y recolección de datos. El análisis de las discusiones mantenidas en grupos pequeños sobre dos tareas exploratorias se basa en un marco que aborda capas de conocimiento, con respecto a la comprensión del concepto de integrales definidas. Como resultado, pudimos inferir que con las tareas exploratorias los estudiantes lograron explorar sustancialmente el concepto de suma base multiplicativa presente en la suma de Riemann, en relación con las capas producto, suma y límite.

Palabras clave: Enseñanza de Cálculo Diferencial e Integral, Integrales de Riemann, Capas de conocimiento, Episodios de resolución de tareas.

Résumé

Compte tenu des difficultés qu'éprouvent les étudiants de la discipline Calcul différentiel et intégral à comprendre le concept d'intégrale définie d'une variable, cet article vise à étudier l'élaboration et la mise en œuvre d'une proposition d'intervention, basée sur le travail avec des épisodes de résolution de problèmes. Offre aux étudiants la possibilité d'explorer ce concept. Nous discutons, comme fondement théorique, de l'importance des intégrales de Riemann et des sommes de base multiplicatives dans la compréhension des intégrales définies. Nous avons également fourni une caractérisation de la méthodologie utilisée dans notre recherche, ainsi que le contexte d'intervention et de collecte de données. L'analyse des discussions tenues en petits groupes autour de deux tâches exploratoires s'appuie sur un cadre qui traite des couches de connaissances, en ce qui concerne la compréhension du concept d'intégrales définies. En conséquence, nous avons pu déduire qu'avec les tâches exploratoires, les étudiants ont pu explorer de manière approfondie le concept de somme de base multiplicative présent dans la somme de Riemann, en relation avec les couches produit, somme et limite.

Mots-clés : Enseignement du calcul différentiel et intégral, intégrales de Riemann, couches de connaissances, épisodes de résolution de tâches.

Resumo

Considerando as dificuldades que alunos na disciplina de Cálculo Diferencial e Integral têm em compreender o conceito de integral definida de uma variável, este artigo tem por objetivo investigar a elaboração e a implementação de uma proposta de intervenção, a partir do trabalho com episódios de resolução de tarefas, que ofereça aos estudantes oportunidades para explorar esse conceito. Discutimos, como fundamentação teórica, a importância das integrais de Riemann e somas de base multiplicativa na compreensão das integrais definidas. Apresentamos uma caracterização da metodologia utilizada em nossa pesquisa, assim como do contexto de intervenção e coleta de dados. A análise das discussões realizadas em pequenos grupos acerca de duas tarefas exploratórias é baseada em um referencial que trata das camadas do conhecimento, no que diz respeito à compreensão do conceito de integrais definidas. Como resultados, pudemos inferir que com as tarefas exploratórias os estudantes puderam explorar substancialmente o conceito de soma de base multiplicativa presente na soma de Riemann, em relação às camadas do produto, da soma e do limite.

Palavras-chave: Ensino de Cálculo Diferencial e Integral, Integrais de Riemann, Camadas do conhecimento, Episódios de resolução de tarefas.

Definite integrals of a single variable: a proposal for intervention using exploratory tasks.

The subject of Differential and Integral Calculus (DIC), both nationally and internationally, has recorded high failure and dropout rates for several decades. These rates are prevalent not only in traditionally “hard” disciplines such as Mathematics and Physics, but also in Engineering courses (Silva, 2013; Zarpelon, 2022). One of the reasons for such high dropout rates is related to *how* the DIC course is typically approached at universities.

> Traditionally, mathematics courses both in Basic Education and especially in Higher Education are “Cartesian,” grounded in the triad of definition-example-exercise, followed by a cumulative assessment that emphasizes the reproduction of procedures. One of the effects of this tradition is that these subjects, from the students' perspective, become tedious and algorithmic, lacking clear objectives and motivation particularly within engineering programs (Couto, Fonseca & Trevisan, 2017, p. 51).

We argue that teaching and learning environments that diverge from traditional methods are necessary in Higher Education mathematics courses, as they foster student protagonism and enhance the understanding of mathematical concepts (Trevisan & Mendes, 2018; Trevisan, Alves & Negrini, 2021; Trevisan, 2022). We align with Souza and Fonseca (2017, p. 198), who state that the teaching of mathematics courses at the Higher Education level, “in its broad contribution to individual development, should be linked to methodological approaches that support the tackling of real-world problems that will be part of students’ future professional practice.”

One possible approach in this direction involves the use of exploratory tasks (Ponte, 2005), which are more open-ended in nature and can be addressed intuitively, even before a formal definition is presented. This encourages students to think autonomously and, through the teacher's interventions, to explore mathematical concepts rather than merely reproduce them. This method also emphasizes collaborative work (Granberg & Olsson, 2015), fostering peer interaction in which one student’s input can influence and potentially redirect another’s line of reasoning when the latter is on an unproductive path.

Regarding the concept of the Riemann Integral through Riemann Sums—the focus of this study—its understanding is essential for addressing problems not only within Mathematics, but especially in subsequent fields of Science (Haddad, 2013). Nevertheless, many students, even after completing the DIC course, lack a solid understanding of this concept. They often view Riemann sums merely as an initial procedure for approximating the value of a definite integral, which is later replaced by the strategy of computing definite integrals through antiderivatives, in conjunction with the presentation of the Fundamental Theorem of Calculus.

According to Jones, Lim, and Chandler (2017, p. 1076).

> Yet several recent studies have clearly demonstrated that calculus students often have difficulties using the concept of integration both in mathematics coursework and in subsequent science coursework. [...] In studying these student difficulties and the reasons for them, there is a recognition that the ideas contained in the Riemann sum structure are important for a robust understanding of definite integration.

This article, an excerpt from the first author's master's thesis, aims to investigate the design and implementation of an intervention proposal based on the analysis of task-solving episodes, which provides students in Differential and Integral Calculus with opportunities to explore the concept of a single-variable integral. To achieve this objective, the theoretical foundation for the intervention was grounded in the concepts of layers of knowledge associated with the concept of the definite integral (Sealey, 2006, 2014), linked to the notion of Multiplicative Base Sums (Jones, Lim & Chandler, 2017), which will be detailed in the following section.

Theoretical Framework

The definite integral is a fundamental concept in the Differential and Integral Calculus course, employed in solving a wide range of problems in both Mathematics and other scientific fields, including Engineering. However, according to Jones (2013, p. 123, our translation), "researchers have noted a perception among educators that students transitioning into science courses are routinely struggling to apply their mathematical knowledge to the domain of science." For the author, students have difficulty employing the concept of the definite integral both within mathematics itself and in applied science contexts, as it possesses multiple facets in its structure and many forms of interpretation.

> Some reasons for these difficulties may be an overreliance on algebraic representations of function (Rubio & Gómez-Chacón, 2011), a lack of understanding of the Riemann sum structure (Orton, 1983; Sealey, 2014), a poor understanding of rates of change and accumulation (Thompson, 1994; Thompson & Silverman, 2008), and struggles in objectifying accumulation as a function (Kouropatov & Dreyfus, 2014; Swidan & Yerushalmy,2014; Yerushalmy & Swidan, 2012). (Jones, Lim & Chandler, 2017, p. 1076)

Sealey (2006) highlights in her work several reasons that make the study of the Riemann Sum substantial for a robust understanding of definite integrals. One of these reasons is the fact that "[...] many real world applications involve functions that do not have an antiderivative that can be expressed in terms of elementary functions" (Sealey, 2006, p. 46, our translation).

Another reason is that, although the Riemann Sum is not the fastest method for approximating the value of a definite integral,

> [...] other methods, such as the trapezoid rule, midpoint rule, or Simpson's method are based on the structure of the Riemann sum. Thus, an understanding of the structure of Riemann sums will help students to understand these other methods as well. Finally, I hypothesize that an understanding of Riemann sums is needed even when a function has an antiderivative that *can* be expressed in terms of elementary functions. Setting up the appropriate definite integral requires the student to know what to integrate, and an understanding of the structure of the Riemann sum will give the student the tools he/she needs. (Sealey, 2006, p. 46).

Still on this topic, Jones, Lim, and Chandler (2017, p. 1076, our translation) emphasize that, when "In studying these student difficulties and the reasons for them, there is a recognition that the ideas contained in the Riemann sum structure are important for a robust understanding of definite integration (Artigue, 1991; Blomhøj & Kjeldsen, 2007; Sealey, 2014)." This suggests that merely performing algebraic manipulations and "solving" a definite integral is not sufficient for students to effectively use integrals as a problem-solving tool. A solid understanding of the multiplicatively based summation structure underlying the Riemann sum is essential. A Multiplicatively Based Summation (MBS) is defined by two core components:

> (1) the multiplicative relationship between the integrand and the differential to create a resultant product, and (2) the idea of summing up small amounts (possibly infinitesimally small) of the resulting product throughout small pieces of the domain (possibly infinitesimally small) to capture the total amount of that quantity. (Jones, Lim & Chandler, 2017, p. 1076).
> De forma mais detalhada, Jones (2015) discute uma MBS a partir da
> This conception contains the idea of breaking the domain into infinitely many, infinitesimally-small sections, each represented by the differential, "dx" [...] A key component of this conception is that a single piece, called a representative rectangle (for the single-variable case), is used to conceptualize the multiplicative relationship between the quantities represented by the integrand and the differential. This product produces an infinitesimally tiny amount of a resultant quantity. The resultant quantity is then added up (or accumulated) across all the infinitely many sections in the domain to get the total amount. (Jones, 2015, p. 156)

Similarly, Thompson and Silverman (2008, p. 4) emphasize that a "Riemann sum, then, made by a sum of incremental bits each of which is made multiplicatively of two quantities, represents a total amount of the derived quantity whose bits are defined by $f(c).\Delta x, c \in [x, x + \Delta x]$". The authors present an example within the context of calculating the work done by a force over a displacement interval to illustrate this idea.

> To understand the idea of accomplished work, for example, as accruing incrementally means that one must think of each momentary total amount of work as the sum of past increments, and of every additional incremental bit of work as being composed of a force applied through a distance. (Thompson & Silverman, 2008, pp. 1)

Thus, if we consider a variable force during displacement, the value of the work done by this force can initially be approximated by the sum of the product of force and displacement (an MBS), considering displacement subintervals that are infinitesimally small. This idea is also present in various other contexts, such as in the calculation of the mass of a rod, in which we assume that, to calculate the mass over infinitesimally small subintervals, we multiply a representative density in each subinterval by the length of the subinterval. This relationship is used to conceptualize the multiplicative connection between the quantities represented by the integrand and the differential. This product yields an infinitesimally small amount of resulting mass. The resulting mass is then summed (or accumulated) across all the infinite subintervals of the domain to obtain the total mass of the rod.

With the aim of investigating how Calculus students understand an MBS, Sealey (2014) studied their engagement with problems situated in physical contexts and proposed a four-layer framework. The idea of layers of knowledge is based on the decomposition of the Riemann Integral, as presented in Sealey's work (2008, 2014). The author proposes a framework (see Table 1) organized into four layers corresponding to the operations involved in calculating a Riemann Integral, namely: product, summation, limit, and function.

Table 1.

Layers in the Structure of a Riemann Integral (Adaptado de Sealey, 2014, p. 234)

Layer	Symbolic representation
Product Layer	$\left[\frac{1}{c} \cdot f(x_i)\right] \cdot [c \cdot \Delta x]$
Summation Layer	$\sum_{i=1}^{n} f(x_i)\Delta x$
Limit Layer	$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
Limit Layer	$f(b) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

The first layer of the Riemann Integral Framework, the Product Layer, "is composed of the multiplication of twoquantities, $f(x_i)$ and Δx, where $f(x_i)$ may be conceptualized as a rate and Δx as a difference" (Sealey, 2014, p. 231). The Summation Layer includes adding all the infinitesimal pieces formed by the product of $f(x_i)$ and Δx, that is, $\sum_{i=1}^{n} f(x_i)\Delta x$. The third layer, in turn, involves the limit "[…] as n approaches infinity of the previous two layers, giving us the Riemann integral." (Sealey, 2014, p. 231).

Finally, "the fourth layer allows one to consider the definite integral as a function where the input is the upper limit (i.e. rightendpoint) of the interval over which the function is integrated, and the output is the numerical value of the definite integral" (Sealey, 2014, p. 232).

Although this is not trivial, understanding the structure of the integral based on the accumulation function may create opportunities for students to learn integration in a meaningful way and to apply it substantively. For that to occur, students need "(a) to see the area under a curve as representing the accumulation of quantity and not necessarily an area, and (b) to make the connection between the derivative and the integral" (Swidan & Yerushalmy, 2016, p. 31).

Assumptions and Intervention Planning

This study is part of a sequence of investigations into the teaching of Differential and Integral Calculus (DIC) from the perspective of working with episodes of exploratory task resolution, developed within the research group to which the authors belong (Trevisan & Mendes, 2018; Trevisan, Alves & Negrini, 2021; Trevisan, 2022).

We draw on the definition of task proposed by Ponte (2005, p. 1):

> A task is, therefore, the objective of the activity. The task may arise in various ways: it can be formulated by the teacher and proposed to the student, stem from the student's own initiative, or even result from negotiation between the teacher and the student. Moreover, the task can be explicitly stated at the beginning of the activity or may emerge implicitly as the activity unfolds.

According to the author, a task can be characterized in several ways, and its structure will depend on the teacher's objective. In contrast to closed-structure tasks (commonly referred to as exercises), exploratory tasks are open-ended and carry a moderate level of challenge. Working with this type of task is "more effective, in terms of learning, when students discover their own method for solving a problem than when they are expected to learn the teacher's method and recognize how to apply it in a given situation" (Ponte, 2005, p. 9).

This approach is supported by the Brazilian National Curriculum Guidelines for Undergraduate Engineering Programs (Brasil, 2019), which emphasize the need to foster the development of general competencies throughout the educational process. One of these includes:

> analyzing and understanding physical and chemical phenomena through symbolic, physical, and other models, verified and validated by experimentation:
> a) being able to model physical and chemical phenomena and systems using mathematical, statistical, computational, and simulation tools, among others;
> b) predicting the behavior of systems using models;
> c) designing experiments that generate real results regarding the behavior of the phenomena and systems under study. (Brasil, 2019, p. 2)

Accordingly, in an undergraduate Engineering program, it is expected that a student enrolled in DIC will be able to use the Mathematics studied in this course to model solutions in the context of other sciences, including physical and chemical phenomena. Thus, it reinforces the importance of planning an intervention that genuinely enables students to understand DIC concepts, establishing the necessary relationships and conjectures to solve problems within their academic context and, in the future, in their professional practice.

Learning, in this context, takes on a collaborative character, as it "[...] encourages the student's participation in the learning process and makes learning an active and effective process" (Torres, Alcântara & Irala, 2004, p. 131). Collaborative work allows students' hypotheses to be questioned, fostering reflection and enabling their validation through sound reasoning or their refutation if they are inconsistent or incorrect. In this process, peers offer support and assist in the construction of knowledge. Furthermore, this approach contributes to "preparing students more effectively for the challenges encountered outside the school environment" (Torres, Alcântara & Irala, 2004, p. 135).

The teacher's role in classes structured around the resolution of exploratory tasks and collaborative work is to guide students in their discussions so that they can achieve the learning objectives. Therefore, it is essential that the teacher knows their students and designs exploratory tasks that promote peer discussion and learning.

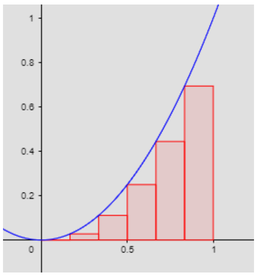
The intervention was planned for Engineering classes in which the first author served as the instructor for DIC II in 2022. Since these students had taken DIC I remotely due to the COVID-19 pandemic, it was observed that it was necessary to revisit the concept of the definite integral of a single variable before advancing to the study of multivariable integrals.

We designed two exploratory tasks, each intended to be solved in small groups during a 50-minute class session. Two additional 50-minute sessions were then used to discuss students' solutions and to systematize the concepts of the meaning and structure of the Riemann Sum and the Riemann Integral.

Exploratory Task 1 aimed to revisit the idea of accumulated area under a curve, summation notation, the structure of a summation, graphical representation of a summation, the calculation of a definite integral, and the relationship between the integral and the calculation of the area beneath a curve, bounded by lines and the $x - axis$. Furthermore, it sought to promote an intuitive understanding associated with the Limit Layer that, as the number of rectangles increases, the sum of their areas approaches the exact value of the area under the curve. This task was inspired by previous work from the research group (Trevisan & Goes, 2016, 2017; Borssoi & Silva, 2020) and was carried out during a 50-minute class session.

Exploratory Task 1

Consider the region bounded by the curve $f(x) = x^2$, , the $x - axis$, and the vertical lines $x = 0$ and $x = 1$.



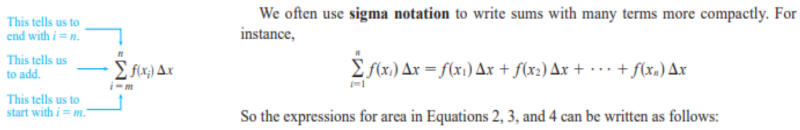
1. Suppose the region is filled with several rectangles, as shown in the adjacent figure, all with the same base.

(a) What is the length of this base?

(b) What is the height of each rectangle?

(c) Estimate the area of the region using these rectangles.

2. Read the definition below.



We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

a) Represent, using summation notation, the calculation performed in item (c) of the previous question.

Use GeoGebra, via the link https://www.geogebra.org/m/PnRNynS8, to create a figure that illustrates $\sum_{i=1}^{8} \frac{1}{8} f(x_i)$ and indicate the value of this sum using a lower sum and then an upper sum. Take a screenshot of your solution and send it via WhatsApp.

3. Determine the exact value of the area of the region using concepts from Calculus.

4. What relationships can you identify between the values obtained in questions (1) and (2) and the value obtained in item (3)?

Exploratory Task 2, designed for a 50-minute class session, aimed to revisit the multiplicative structure present in a Riemann sum through a physical context and, from there, define a single-variable Riemann integral. The task provided students with the opportunity to understand the definite integral as the limit of a multiplicatively based sum, rather than merely as the "area under the curve." In this way, the task enables the mobilization of the summation, product, and limit layers that comprise the conceptual structure of the Riemann integral.

Exploratory Task 2

Comentado [AC1]: Voltar e arrumar

The term work is commonly used in everyday language to refer to the amount of effort required to perform a task. In Physics, however, this term has a technical meaning that depends on the concept of force. Intuitively, you can think of force as describing a push or pull on an object. In general, if an object moves along a straight line with position function $s(t)$, then the force F acting on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration: $F = ma$

In the case of constant acceleration, the force F is also constant, and work is defined as the product of the force F and the distance the object travels, or its change in position Δs:

$W = F \cdot \Delta s$

1. How much work is done when lifting a $1{,}2kg$ book from the floor to a desk $0{,}7m$ high ? Consider that the acceleration due to gravity is $g = 9{,}8m/s^2$.

2. The equation above defines work assuming the force is constant. But what happens if the force is variable? Now imagine a context in which the force $F(x)$ varies along the x-axis in the positive direction from $x = a$ to $x = b$. Even so, we could assume the force is approximately constant within subintervals of $[a, b]$.

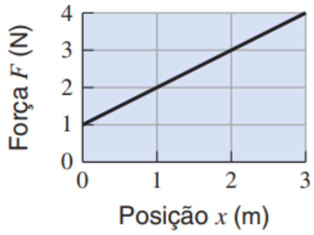
i) For a force function that varies according to the graph below, in each case calculate the work done by the force on a particle that moves from:

a) $x = 0$ to $x = 1$.

b) $x = 1$ to $x = 2$.

c) $x = 2$ to $x = 3$.

d) $x = 0$ to $x = 3$.



ii) Represente cada um dos cálculos realizados no item anterior em notação de somatório.

iii) Que relações você estabelece entre os cálculos anteriores e a tarefa da aula passada?

3. Suponha que um objeto se mova no sentido positivo, ao longo de um eixo coordenado, sujeito a uma força variável $F(x)$ que é aplicada no sentido do movimento. Proponha um método que permita calcular o trabalho W realizado por essa força, quando o objeto se move de $x = a$ até $x = b$.

Procedures methodological

We recognize that this study aimed to investigate a specific reality and, through this investigation, to gather sufficient elements to design an intervention within that context. This allows us to characterize it as a qualitative study (Bogdan & Biklen, 1994).

According to Gerhardt and Silveira (2009), qualitative research is not concerned with numerical representations of the studied context, but rather with a deep understanding of a social group or an organization. For these authors, researchers who adopt qualitative methods "seek to explain the why of things, expressing what should be done, but do not quantify values or symbolic exchanges, nor are they bound by empirical verification" (Gerhardt & Silveira, 2009, p. 32).

Data collection was conducted in two classes during the second half of the first semester of 2022 — one with 35 enrolled students and the other with 15. The exploratory tasks were carried out in small groups of 4 or 5 students, promoting interaction among participants. All students were informed about the research being conducted and were given the option to consent or not to the use of the material collected during class for analytical purposes. It is important to note that the written protocols submitted by the groups at the end of each task were already part of the course routine, as outlined in the syllabus, and were used to support ongoing assessment throughout the semester. Thus, it was the students' responsibility to record the group task discussions and submit them via WhatsApp, authorizing their use for research purposes.

The first step involved organizing the audio files in Google Drive and selecting those that could be effectively analyzed, discarding any that were inaudible or where the group simply summarized their solution at the end of the task, rather than recording the full discussion process as requested by the researcher.

Out of the two exploratory tasks, a total of 10 audio recordings were collected. Of these, only 4 were both audible and suitable for analysis. One group was selected, considered representative for analysis, composed of students here referred to as Student 1, Student 2, Student 3, and Student 4. Excerpts were chosen that allowed the identification of ideas and understandings regarding the structure of a definite integral.

These audio excerpts were fully transcribed and cross-referenced with the corresponding written protocols. Then, the data analysis began, identifying, in each segment of the transcribed speech, the layers of the Riemann sum structure (Sealey, 2014; Jones, 2015). The presentation and analysis of these findings are provided in the following section.

Data Analysis and Discussion

In this chapter, we present and analyze data resulting from the implementation of the proposed intervention, based on episodes of task resolution related to the two exploratory tasks. Through the data presented, we aim to highlight opportunities to (re)conceptualize the notion of the definite integral of a single variable based on Tasks 1 and 2. For better reader comprehension, excerpts of interest are presented in **bold**, followed by analysis and reflections regarding the highlighted segment.

Exploratory Task 1

The group begins the discussion of Question 1, which involves the region bounded by the curve $f(x) = x^2$, the $x - axis$, and the vertical lines $x = 0$ e $x = 1$. The region is filled with a number of rectangles, and students are asked to identify the width and height of each rectangle, and then calculate an approximate area of the region using those rectangles.

Student 3: Okay, we need to figure out if the base is the total one or just the rectangle's.

> Student 4: Let's do both, because if we figure out the rectangle's base, we can kinda estimate the total base.
> Student 2: For the rectangle's base, we'd take 0.5 and divide it by 3.
> Student 1: The problem is, it's a curve. If it were a rectangle…
> Student 2: It'd be easy.
> Student 1: Base times height divided by 2.
> Student 2: No, with a rectangle you don't divide by 2.
> Student 4: Yeah, it's just base times height.
> Student 1: No, I mean like, if it were a straight line, we'd use the rectangle, divide by 2, and it'd work.
> Student 3: We gotta do it.

In this first excerpt of the dialogue, the group explores one of the elements related to the product layer, which is "composed by the multiplication of two quantities,$f(x_i)$ and x, Where $f(x_i)$ can be conceptualized as a rate and x a difference" (Sealey, 2014, p.); n this case, the understanding of the base of the rectangles that fill the region delimited by the curve and the

Comentado [AC2]: Comecei aqui.

axis. In the image presented in the task, there was a marking of a point with abscissa 0,5 as the midpoint of the interval [0,1]. Thus, from Student 2's statement, we understand that he recognizes the base of the rectangles will be one third of that value or, equivalently, one sixth of the interval. Meanwhile, Student 4 points out that the area of each rectangle will be obtained by the product of the base measure and the height.

> Student 1: Maybe it's an integral?
> Student 2: No, it's gonna be an integral using Riemann sums. Like, we take the amount of area intervals underneath and divide by it, like first times... the first divided by the total. It's 1 minus 0.
> Student 1: So the base is gonna be...
> Student 3: Over 5, right?
> Student 2: Yeah, over 5.
> Student 4: 'Cause the way I see it, you could just do a definite integral from 0 to 1 of the function x^2.
> Student 2: Yeah, but that's basically what Riemann sum is a definite integral from a to b.
> Student 4: Which would be from 0 to 1, right?
> Student 2: Uh-huh. And then the limit goes to infinity, or in this case, goes to 5 of the sum, like the summation of…
> Student 2: Wait, why is it going to 5?
> Student 4: Cause we have 5 subintervals.

om the continuation of the dialogue, we can note that Student 2 refers to the Riemann sum to determine the area under the curve $f(x) = x^2$. He highlights elements from both the sum layer and the limit layer, in which, as nnn approaches infinity, from the two layers—sum and product—emerges the Riemann integral (Sealey, 2014), by stating that a definite integral is the limit of a Riemann sum. In turn, Student 4 recognizes that this area could be calculated exactly through a definite integral. However, it seems that the understanding of conceptual issues concerning these layers is not clear to them — for example, when Student 2 says that there should be a limit approaching 5.

> Student 2: Hmm, okay. I hadn't thought about that.
> Student 3: x^2dx.
> Student 4: Then when you derivative x, not when you integrate x^2, you get x over 3.
> Student 3: x^3 over 3.
> Student 2: You'll take 0 and 1.
> Student 4: What do you mean?
> Student 2: Substituting 0 gives 0. Substituting 1 gives $\frac{1}{3}$
> Student 4: No, but when you integrate...
> Student 1: First you do the indefinite integral.
> Student 3: You'd have to integrate $f(x)$, limited by 0 and 1, to find the [inaudible] of the base.

Student 2: What do you think?
Student 4: No, I think if we do the integral, we'll get the total area, right? But he just wants the area of the rectangles.

The group uses the procedure for calculating a definite integral by determining the antiderivative, even though that was not what had been requested at that stage of the task. Thus, after finding the value of the area under the curve using a definite integral, Student 4 draws the group's attention to what the task statement actually said. They then begin to find the base of each rectangle, that is, Δx.

Student 3: Yeah.
Student 2: We're not even talking about the area yet, right? Just the base?
Student 4: It's the base, look.
Student 2: Hold on, let me divide.
Student 4: It's 0.5 divided by 1, right? Or by 3 or 2, doesn't really matter.
Student 2: So, each interval is 0.16.
Student 4: 0.16?
Student 2: Yeah, 0.16666.
Student 1: Does he want the area of each rectangle?
Student 2: Yep.
Student 4: The area? He just wants the height later, right now he's asking for the base.
Student 3: We'll probably only integrate in item (c), which is about calculating the approximate area of the region.
Student 4: Yeah, then we'll use the sum thing... I forgot the name [referring to the Riemann Sum].
Student 2: Yeah, he wants to know the value of the base.
Student 4: Right.
Student 1: Dude, there's 1, 2, 3, 4, 5, 6 - 6 intervals, so you take 1 and divide it by 6. That's 0.166666.
Student 4: Six intervals?
Student 1: Yep.
Student 4: Could it really be that simple?
Student 2: Yeah, we've gotten used to overcomplicating stuff.
Student 1: To me, that's it. What do you think?
Student 2: I think that's it.
Student 4: 0.16666.
Student 1: So item (a) is just that?

In this part of the discussion, the group explores, in fact, what was requested in item (a), that is, to determine the base of each rectangle. This is an important element for understanding the sum layer (Sealey, 2014). Initially, there is doubt about whether to divide the 0,5 measurement into two or three parts, as highlighted by Student 4. After some initial conjectures, Student 1 points out that it is necessary to divide 1 by 6.

Student 4: Yeah, so now we need to figure out the height of each, each rectangle. Lend me the ruler, I'm gonna guess.

Comentado [AC3]: Aqui eu acho que tem que arrumar a formatação

Student 2: Nah, just plug it into the equation $f(x) = x^2$.
Student 3: Yeah, it's $f(x) = x^2$.
Student 4: Oh right, that makes sense!
Student 4: Wait, do we take the end or the start of the rectangle? The end, right?
Student 2: The end.
Student 1: No, it's the start.
Student 4: I'll go with the start then. So the first rectangle starts at 0, it's not gonna have any height.
Student 2: Yep.
Student 4: Then it's gonna be 1.6^2?
Student 1: Wait, are you doing like 1.6 times 1.6^2?
Student 4: No, I'm just being dumb. I said 1.6, that makes no sense. It's 0.16.
Student 1: Which we square. Then next would be 0.37^2.
Student 4: It's 0.32, right?
Student 1: Actually, it would be 0.33, right?
Student 2: Just leave it as a fraction $\frac{1}{6}$.
Student 1: So we'll have to do six calculations, right?
Student 4: Yep.

Student 4 then suggests determining the height of the rectangles using a ruler, in this case, the figure that accompanies the task. However, this suggestion is soon discarded, as Student 2 acknowledges that the value sought is obtained from the expression of the function. This refers to another important element of the product layer (Sealey, 2014), the recognition of the function's value at a point as one of the factors that make up MBS (Jones, 2015).

As the discussion continues, sample points from the interval are observed, and they recognize that the first will be 0, the second 0,16, the third 0,32 (or 0,33). There is doubt about whether to choose the initial or final point of each subinterval, and Student 4 chooses to work with the initial point. However, it does not seem clear to the group why this choice was made (in fact, the task image suggests a filling of the region with rectangles that lie below the curve).

Considering the difficulty in working with approximations, Student 2 suggests using fractional representation. The complete solution to items (a) and (b), systematized in the group's written protocol, is shown in Figure 1.

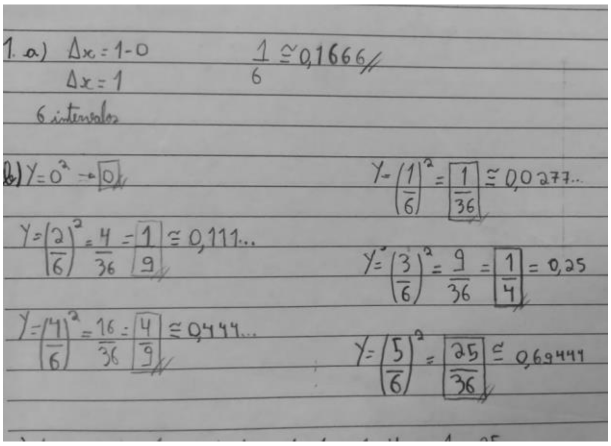


Figure 1.

Solution to Question 1, items (a) and (b) of Investigative Task 1 (research data, 2022)

The group continues the discussion, now referring to item (c) of the task, which asked for the approximate area of the region based on the rectangles.

Student 2: I'll start thinking about item (c), then.
Student 4: But for item (c), you're going to need to use... You're going to use the formula, then.
Student 3: What formula?
Student 2: Well, we have it.
Student 1: Or not.
Student 1: If we have the height and the base, we just multiply one by the other.
Student 4: Isn't it divided by 2?
Student 1: I'm not talking about dividing by 2, because like, we're going to have the base and height of each one, multiply one by the other, and then just add them.
Student 4: But then there's going to be a piece left over, right? Each one will have something left.
Student 1: No, but what we want is the region.
Student 3: The region bounded by a curve. Then it would have to be calculated.
Student 4: Approximate the area of the region using rectangles.
Student 1: She wants it approximated, so it's rectangles for sure.
Student 3: No, you have to use the rectangles over the area that the curve bounds.
Student 2: Approximate the area of the region using these rectangles—if it's approximate, it's not to calculate the exact area under the curve.
Student 3: I think we have to calculate the area under the curve up to 1, you know? Then calculate the area.
Student 2: Because if he wants the approximate area, then it probably wasn't... in this case, it's going to be this [pointing at the drawing]. But he said to use the rectangles.
Student 3: Yeah, but I think the rectangles are used as a base to find the area in item 3, like, I think that's it.

Student 2: Item (c).
Student 3: Like, there are two options here: either we find the total area by summing the rectangles or we find the area bounded by the curve, like from the curve down.
Student 2: No, but look, he says: using the rectangles.
Student 3: So, that complicates things. Because then I don't know if you're supposed to use the rectangles to bound the curve and calculate the area or if you're supposed to sum the areas of the rectangles.
Student 1: Actually, the idea behind the integral is that you can calculate the area, so I think, when he gave us this, it doesn't make sense for us to use any method other than the integral.
Student 4: Actually, he's bringing it back to help us recall the integral, right?
Student 3: Yeah, this is like a concept of the integral, you know? Not necessarily the integral itself.
Student 4: He told us to use the rectangles.
Student 2: He said to use rectangles and that it's approximate too.
Student 1: Since he's telling us to use the rectangles, we'll just sum the rectangles and round up.

From Student 1's initial statement, we see that the group recognizes the multiplicative structure present in the calculation of area, since it is necessary to multiply the base measurement by the height measurement. However, the group is unsure whether this procedure is appropriate, given that the rectangles do not fully fill the region and, as highlighted by Student 4, a piece will be "left over" in each rectangle. The students discuss the inefficiency of using the area of rectangles to approximate the total area, since they are aware of the definite integral as a tool that provides an exact value for the area under the curve.

At the end of this excerpt, the group acknowledges that this is a task meant to "recall" the concept of integrals, and they carry out the calculation shown in Figure 2.

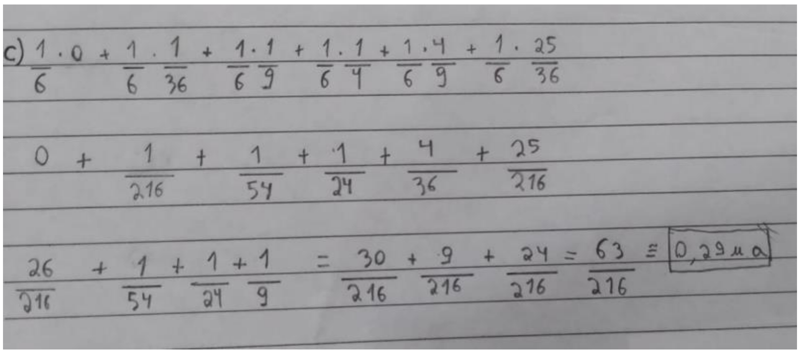


Figure 2.

Solution to Question 1, item (c), of Investigative Task 1 (research data, 2022)

The group then moves on to Question 2, which, after a brief explanation about summation notation, asks that it be used to represent the previously performed calculation.

Student 4: A convenient way to write it (...) capital sigma.
Student 2: Where's the general formula?
Student 4: Summation notation.
Student 1: It's this here, but like, this is a Riemann sum, this is just the sum.
Student 2: But that's what I wanted.
Student 1: So, it's sigma.
Student 4: You just do the summation, from $i = 1$, then we write a equals, a of 1, a of 2 plus a of 3, right?
Student 2: What do you mean? No, you have to take.
Student 4: Sum of a, from 1 to 6.
Student 2: Sum of the first, the second, the third.
Student 4: So, we do the sum of the areas from 1 to 6, right?
Student 2: Yeah, but how are we going to write that?
Student 4: Summation from n, $i = 1$, $a_1 + \ldots + a_6$.
Student 2: Why $i = 1$?
Student 4: Oh, I always write $i = 1$ and it's always right — why? I don't know.
Student 1: Because i goes from 1 to the number you put on top.
Student 3: But in this case, we're considering 6, right? We would have to start from here below.
Student 1: So 6 goes on top.
Student 3: We'd have to start from down here.
Student 1: Exactly.
Student 3: So i is the starting number.
Student 1: No, for example, here I did: $f(x_i)$, and i goes from 1 to 6. $f(x_1)$ what is x_1?
Student 4: It's the y when x equals 1, isn't it?
Student 3: That's right.
Student 1: No, when x equals x_1, x_1 here is 0.
Student 4: So x_i here will vary, it will go from 0 to...
Student 1: Yeah, we have to think of a good way to write this formula, one that looks nice. Let me think… a way where the first is 0, you get it?
Student 2: It's the summation of the first one, if i starts at 0, if $i = 0$, then it's $a + 0$ which equals zero.
Student 1: Summation from $i = 1$ to 6 of a_i, ok?
Student 2: Cool.
Student 1: Which equals, now let's expand this a. 6 equals 1, $f(x_i)$.
Student 4: So, what will A be then, guys?
Student 1: Man, we haven't reached a consensus.
Student 2: Dude, x_i gives a, how are we going to write the formula?
Student 1: We'd have to start from 0, so it would be the sum of $0 + 0$.
Student 4: Can I write what I think it will be?
Student 3: Isn't it just the area of the first rectangle plus the second, plus the third, plus the fourth up to the sixth, and then sum it to get the value?
Student 2: Exactly.
Student 4: I guess it would be something like this well, not guessing, I think it is.
Student 2: Yeah.
Student 1: Yeah, just leave it in summation form, a_i.
Student 4: That works. Easier, right? So how do we write it?
Student 1: However you want.
Student 4: Then I'll write it out in full, you know?

Student 4: Ok, so how should we write it? Like it's in your notebook or the way I wrote it?
Student 2: Ah, go with the one in the notebook.
Student 3: Summation.
Student 1: Summation from $i = 6$, a_i.
Student 4: $i = 1$ to 6, a_i.

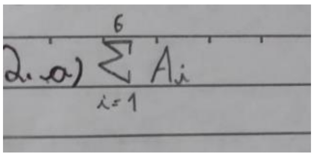


Figure 3.

Solution to Question 2, item (a), of Investigative Task 1 (research data, 2022).

In this transcript excerpt, the students try to understand the structure of summation and apply it to the situation under analysis. The group reaches a consensus regarding the variation of the index, in this case from 1 to 6 (Figure 3). However, it is not clear to them how to represent the internal term of the summation. This discussion on summation, although with some incomplete and this discussion on summation, although with some incomplete and incorrect conclusions, was important for the group to later be able to understand the representation of the sum of infinite pieces connected to the concept of definite integrals for one or more variables. Moreover, in a later moment of systematization, the teacher helped the class organize this summation structure to represent MBS (Jones, 2015).

Finally, the group discusses Questions 3 and 4, which request the exact value of the area using a definite integral and the establishment of relationships between that value and the one obtained in Question 1.

Student 2: Knowledge of calculus, it'll give you the exact value.
Student 4: **So what you do is, you get x^2, then $\frac{x^3}{3}$.**
Student 2: Right.
Student 4: Like, if you plug in **1 for x, becomes $\frac{1}{3}$, and if you take $0,333$ as in our item (c), that 0,29 we got, we can say we got pretty close using the integral rather than this way. If you want, I can write it up neatly, if you all agree that's it**.
Student 2: Nah, yeah, I agree.
Aluno 4: What do you think? Like this here. You integrate it, then it becomes $\frac{1}{3}$, que dá 0,333 which gives 0,333 and if that's different from what we got, it's because there are these little areas we didn't count.
Student 1: Yeah, but how do you know the integral of $\frac{x^3}{3}$ is $\frac{1}{3}$?
Student 2: By the rule.

Student 1: Yeah, but I don't really remember how it goes.
Student 2: The derivative rule is like...
Aluno 4: It's like this — if you integrate at 0, you get 0, so there's no area yet. Then you integrate at, hen you integrate at 1, and it gives you $\frac{1}{3}$ if you plug 1 into x^3.
Student 2: No, what I wanna know is how we got to $\frac{x^3}{3}$ in the first place, right?
Student 1: Oh, yeah.
Student 4: So, did you integrate x^2?.
Student 2: No, but that's the thing, I wanna know the process, we just knew it had to give that result (...).
Student 4: I didn't get it.
Student 2: There's a process that takes you from x^2 to x^3.
Student 4: Ohhh.
Student 2: Like, I know how to explain it — when you derive it, you don't take the exponent, you decrease it by one and that's it.
Student 1: You "drop the power."
Student 2: If we think of it like that, the integral is the opposite — like, if we increase the power, we gotta add one to the numerator, and since there's nothing on the bottom, something's gotta go there or it wouldn't be 3.
Student 3: The opposite of multiplying is dividing, so you reduce the exponent. It's just like the derivation rule. If you all think that's it, I can write it out.
Student 1: Sounds good to me.
Student 2: So, the only thing missing is this sign here, $\frac{x^3}{3}$ but you gotta put that in.
Student 3: This part here — is it needed? Because here you didn't integrate, you just solved it(...).
Student 2: Nah, nah, here you're just doing $\frac{x^3}{3}$ not integrating.
Student 3: Right, this is just evaluating it at 1 and 0.
Student 2: That's what you're gonna use later.
Student 3: So here it's just 0 and 1, right? What did you plug in?

We can note, in this excerpt, that the students relate the antiderivative as the inverse process of the derivative of a function, using it to calculate the definite integral. The group seems to handle algorithmic questions well, but without making explicit their understanding of the relationship between the exact value calculated in this way and the one previously obtained. In Figure 4, we have the answer provided by the group, containing some important elements from the limit layer.

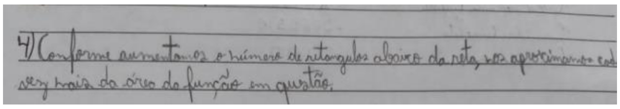

4) Conforme aumentamos o número de retângulos abaixo da reta, nos aproximamos cad
vez mais da área da função em questão.

Figure 4.

Solution to Question 4 of Investigative Task 1 (research data, 2022).

Summarizing the analysis of this task, we infer some opportunities to explore the concept of the definite integral of a single-variable function that the task offered, especially by activating some elements from the product, sum, and limit layers (Sealey, 2014), as highlighted in Table 2.

Table **2**

Opportunities to explore the concept of the definite integral of a single-variable function (the authors).

Comentado [AC4]: Acho que deve arurmar, ficou em 2 linhas

Product Layer	Recognition of the measure of the base of each rectangle; Determination of the abscissa of the partition points of the interval; Recognition of the measure of the rectangle's height as the value of the function at the initial point of each subinterval.
Sum Layer	Determination of the approximate area through a summation process; Understanding of the variation of indices in summation notation.
Limit Layer	Intuitive idea that, as the number of rectangles increases, the sum gets increasingly closer to the exact value of the area.

Thus, the task provided opportunities for the members of this group to activate their knowledge, even if incompletely, to explore elements of the layers that make up the structure of a Riemann integral. Furthermore, it allowed them to later take a more active role in the collective discussion led by the researcher to systematize the structure of the concept of multiplicative base sums in the definition of a definite integral.

Exploratory Task 2

In this task, we focus on the analysis of excerpts from the discussion in which students are encouraged to reflect on the calculation of work in the case of a variable force. To that end, at first, they are asked to calculate, based on a graph of force versus position, the work done in subintervals.

Student 2: What is that? An integral from 0 to 1?
Student 1: Bro, you can just do that with regular math, no need for an integral.
Student 2: Do you make the equation of the line?
Student 1: Nah man, he wants the work, right?
Student 2: So, this one's easy, but...
Student 1: You just find the area, right?

Student 2: Yeah.
Student 1: That's totally doable, bro.
Student 2: You can do it here too, like, to double-check the other calculation you're gonna do — you find the equation of the line, then integrate that equation and plug in the values.
Student 1: But then you gotta do that after.

Though the task suggested that students use partitions of the interval, they seem to recognize the definite integral as a tool that provides the exact value of the work. Student 2 proposes that, to calculate the work done, it would be enough to assume the definite integral from 0 to 3, of the force function that varies according to position. The group correctly determines the algebraic expression of this function and performs the calculation of the work using integrals.

However, when faced with item (b) of the task, which required the representation of the calculations using summation notation, the group expressed uncertainty.

Student 4: So, what now?
Student 2: Summation notation.
Student 4: Summation is what we did over there, right?
Student 2: 1 2 3 — you were the one who changed it in the calculation, I don't remember. Is it 3 rectangles? So it's summation from 1 to 3 and then...
Student 2: No, summation, interval from 1 to 3.
Student 4: No, yeah.
Student 4: Summation goes from 1 to 3.
Student 2: And how do you write that in the formula you made?
Student 4: Summation of that integral [...] It's gonna be summation from i = 1 to 3 of the integral.
Student 2: I don't know how to write that down, I think...
Student 4: From 0, right?
Student 2: You gotta know how to write the summation notation.
Student 4: But wait, isn't this one with the integral?
Student 2: It's because we're thinking of the summation of the integral, but once we put the integral, it'll already show the area.

In this excerpt, we observe that although the group knew that the work could be calculated through an integral, they were not able to relate this concept to the representation using summation notation. The researcher's intervention was necessary at that point.

Researcher: The total force there—how did you calculate item (a)?
Student 2: Integral from 0 to 1, of x.
Researcher: Did you do the integral?
Student 2: Was it supposed to be just the simple area?
Researcher: Yes.
Student 2: Geometric figure?

Researcher: Because here it says like this: we could assume the force is approximately constant in subintervals. So, it's saying that in each interval, you need to assume that the force is constant in that stretch...
Student 2: Like, saying it's a rectangle here?
Student 1: It's like having little steps, you know?
Researcher: Exactly.
Student 2: Oh.
Student 1: The force is constant here, and here, and here—the force is constant.
Student 4: Oh right, so we did something else.
Researcher: Yeah, when you used the integral, you assumed that within the interval it was already variable. But here it's asking you to assume that within each interval, it stays constant.

At this point, the researcher directs the students' attention to the wording of the question. Recognizing the structure of MBS (Jones, 2015) in the situation was important for the students to actually understand why a definite integral is used to calculate the work done by a force. Thus, it was important to provide the student with the opportunity to understand the calculation of work as the approximate value of a multiplicative base sum.

From the teacher's intervention, Student 1 seems to recognize some similarity between this situation and the area calculation from Task 1, when he mentions that it's "like a little staircase here." Then, he seems to understand that in this strategy, it is assumed that, in each subinterval, the force is approximately constant, as expressed when he says "the force is constant here, the force is constant here...". The researcher then suggests establishing some relation between this situation and the previous task to represent this idea using summation notation:

Researcher: The summation from the previous activity—how did you guys do it? The rectangles up to that 6 remember? You wrote there: summation from i equals 1 to 6.
Student 2: i over 6 squared plus i over 6, but … no, it's multiplied...
Researcher: How many partitions do you have there?
Student 2: 3, from 1 to 3, summation from 1 to 3.
Researcher: That's right.
Student 2: The thing is, we don't know how to write this area like we did in the equation last time.
Researcher: You can leave it as $f(x)$.
Student 4: Summation of f, something like that**?**
Researcher: Exactly.
Student 2: From 1 to 3, summation of $f(x)$.
Researcher: Yeah, but what are you summing up there? What does each rectangle stand for?
[inaudible]
Researcher: Right, geometrically: area. So area isn't just the function, right? It's a product. What's the function here? It's the height, right? If you take f(x) as the height, what's still missing to calculate the area?

Student 3: Length.
Student 4: Its width.
Researcher: We can call it Δx, which is the change in position here.
Student 2: So this here becomes this? Summation of $f(x)$.
Student 3: From 1 to 3?
Student 2: From 1 to 3.
Student 3: **So what matters is the division of these areas that we took from the graph, since there are 3 rectangles So it's summation from $i = 1$ to 3.**
Aluno 1: **Of x times.**
Aluno 3: $\boldsymbol{\Delta x}$**.**

Based on the researcher's intervention, which highlighted elements from the product and sum layers of the definite integral (Sealey, 2014), the group was able to express, using summation notation, the work done in the case of the force provided in item (a) of the task. In particular, she seeks to systematize with the group the elements present in the summation notation: the interval of variation, the value of the function at sample points, and the size of each partition.

The group then discusses the final item of the task, seeking to more generally express the calculation of the work done by a generic force $f(x)$ over any interval $[a, b]$.

Student 3: No, let's say we already have $f(x)$, like, a variable force $f(x)$ that's applied and all that.

Student 1: Suggest a method that allows us to calculate the work.
Student 2: **Just write the integral** of $f(x)$, like that.
Student 4: That's what I was gonna say.
Student 2: **From a to b**. That's it, just write the integral.
Student 4: No, I think I'm going to write it out in words, like "the method to calculate it.".
Student 2: Just say what it's going to be.
Student 4: Work done.
Student 2: What you're going to write here is going to be the integral of…
Student 4: From a to b, of something.
Aluno 2: from a to b.
Aluno 4: Of x.
Aluno 2: of $\boldsymbol{f(x)dx}$, that's it.
Aluno 4: Yeah.
Student 2: You've gotta write it, if you want to spell it out.
Student 4: I'll write it out, then I'll put it down here.
Student 2: So say it like this: you write **"the integral with limits," "from a to b," "from a to b in terms of x," "of the function $f(x)$"**.

Summarizing the analysis of this task, we infer some opportunities to explore the concept of the definite integral of a single variable that it provided, especially by activating elements from the product and sum layers (Sealey, 2014), as highlighted in the table below:

Table 4

Opportunities to explore the concept of the definite integral of a single variable (the authors)

Product Layer	● Recognition of the element Δx as the displacement interval. ● Recognition of the height of the rectangle as the value of the force function at the initial point of each subinterval. ● Understanding of the multiplicative structure of the expression. $f(x)\Delta x$. ● Relationship between the expression $f(x)\Delta x$ and the area of a rectangle.
Sum Layer	● Determination of the range of variation for the summation; ● Establishment of the relationship between the sum of the work done in each subinterval and the work calculated using integrals.

Thus, although the task, at first, did not provide opportunities for the members of this group to explore elements from the layers that compose the structure of a Riemann integral, the role of the researcher was fundamental. In the discussion, it is possible to see that, after the researcher's intervention, the students were able to configure a summation to calculate the approximate value of the work and, later on—even if in an intuitive and simplified way—relate the summation to the integral used to calculate the exact value of the work. It was up to the teacher and the researcher, during the collective discussion, to help the class organize these ideas—some of which were still unclear—with the goal of offering elements to explore the concept of definite integral.

Final Considerations

As the objective of our research, we outlined the development and implementation of an intervention proposal based on work involving task-solving episodes (Trevisan & Mendes, 2018; Trevisan, Alves & Negrini, 2021; Trevisan, 2022), which offers CDI students opportunities to explore the concept of the definite integral of a single variable. As part of our intervention, we conceived the organization of exploratory tasks with the potential for students to access the four layers and, from there, explore the concept of the definite integral (Sealey, 2006, 2014). Through data collection, we sought to identify whether the formulated tasks indeed had this potential.

Exploratory Task 1 provided students with the opportunity to engage with the product layer, as evidenced in the process of recognizing the base length of each rectangle, in determining the abscissas of the partition points of the interval, and in establishing the height of the rectangle as the value of the function at the initial point of each subinterval. The sum layer is explicit in the determination of the approximate area through a summation process and in the understanding of the variation of indices in summation notation. The limit layer is expressed through the intuitive idea that, as the number of rectangles increases, the sum increasingly approximates the exact value of the area.

Exploratory Task 2 provided students with an understanding of the product layer, particularly in the comprehension of the multiplicative structure of the expression used to calculate the work done by a force over a small interval, and of the sum layer, especially in establishing the relationship between the sum of the work done in each subinterval and the work calculated using integrals. Thus, we can infer that through exploratory tasks 1 and 2, students accessed the concept of multiplicative base sum present in the Riemann sum, which is fundamental to exploring the concept of definite integral.

As a limitation of our proposal, we recognize that the function layer was not spontaneously explored in the group discussions based on the task proposal. However, during the systematization conducted by the teacher and the researcher, this layer was addressed to explain the relationship between the area under the curve and the antiderivative in the calculation of the definite integral, using the idea of an accumulation function. Therefore, this aspect should be considered when adjusting the tasks, in order to provide students with the opportunity to mobilize and understand the function layer that is present in a definite integral.

Referênce

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