EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1

http://dx.doi.org/10.23925/1983-3156.2025v27i3p230-257

Figura geométrica no livro e o PCOC correspondente a mão livre: Uma gestão de códigos para a impressora 3D

Geometric figure in the book and the corresponding freehand PCOC: A code management for the 3D printer

Figura geométrica en el libro y el PCOC correspondiente a mano alzada: Gestión de códigos para la impresora 3D

Figure géométrique dans le livre et le PCOC correspondant à main libre : gestion des codes pour l'imprimante 3D

Afonso Henriques[1]
Universidade Estadual de Santa Cruz (UESC)
Doutorado
http://orcid.org/0000-0002-8783-6008

Rosane Leite Funato[2]
Universidade Estadual de Santa Cruz (UESC)
Mestrado
https://orcid.org/0000-0002-6799-0876

Elisângela Silva Farias[3]
Universidade Estadual de Santa Cruz (UESC)
Mestrado
https://orcid.org/0000-0002-4052-2351

Resumo

As Geometrias Plana, Analítica e Espacial, doravante designadas GEOPAES, enquanto disciplinas de Matemática exigem que os(as) estudantes sejam capazes de visualizar e interpretar figuras geométricas referentes aos saberes matemáticos a ensinar previstos nos livros didáticos. Como auxiliar o(a) Professor(a)[4] no ensino e o(a) estudante na aprendizagem de saberes de GEOPAES, favorecendo a referida visualização e a interpretação no ensino em que esses livros são utilizados? Responder esse tipo de questão é um dos nossos objetivos neste artigo, que apresenta uma modelagem paramétrica e gestão de código de um material didático construído em ambiente computacional, materializado na impressora 3D, tornando-se palpável

[1] henry@uesc.br
[2] rlfunato@uesc.br
[3] esfarias@uesc.br

[4] O autor defende que o termo Professor(a) será e deveria sempre ser escrito com a letra P maiúscula, pois este(a) é um(a) profissional que merece e deve ser respeitado(a) como os outros. O simples gesto de aplicar a letra maiúscula engrandece também a sua personalidade (Henriques, 2019, p. 13)

a “mão livre”. Essa modelagem e gestão são práticas usuais em pesquisas realizadas no GPEMAC[5], no Laboratório de Visualização Matemática (L@VIM) da UESC, e vêm servindo o(a) Professor(a) no ensino e o(a) estudante na aprendizagem matemática, bem em outras disciplinas onde a GEOPAES encontra um habitat nessa instituição. Um PCOC (que se lê, “Peceocê) impresso em 3D pode contribuir na identificação e reconhecimento de elementos geométricos notáveis no modelo, na manipulação e no seu posicionamento em relação ao sistema de coordenadas, na aplicação do procedimento “tomográfico” que examina as intersecções de superfícies que o delimitam, favorecendo a análise de traços e de curvas de níveis, entre outros procedimentos que podem ser utilizados pelo(a) Professor(a) e pelo(a) estudante. O modelo apresentado nesse artigo evidencia um tipo dessas intersecções por consequência do objeto geométrico observado nos livros didáticos.

Palavras-chave: Geometrias, Modelagem paramétrica, *Software maple*, Impressora 3D.

Abstract

Plane, analytical, and spatial geometries, which we refer to as GEOPAES, are mathematical topics that require students to visualize and interpret geometric figures related to the mathematical knowings to teach, as outlined in textbooks. How can we assist Teachers in their teaching and students in learning GEOPAES knowings, favoring such visualization and interpretation in the teaching that utilizes those books? Answering this type of question is one of our objectives in this article, which presents a parametric modeling and code management of a teaching material built in a computational environment, materialized in the 3D printer, becoming tangible by “free hand”. These modeling and management practices are standard in research carried out at GPEMAC, in the Laboratory of Mathematical Visualization (L@VIM) of UESC, and have been serving the Teacher in teaching and the student in learning mathematics, as well as in other subjects where GEOPAES is found in this institution. A 3D-printed PCOC can contribute to the identification and recognition of notable geometric elements in the model, in the manipulation and positioning of the model in relation to the coodinate system, in the application of the “tomographic” procedure that examines the intersections of surfaces that delimit it, favoring the analysis of traces and contour lines, among other procedures that the Teacher and the student can use. The model presented in this article highlights one type of these intersections, resulting from the geometric object observed in the textbooks.

[5] Grupo de Pesquisa em Ensino e Aprendizagem da Matemática em Ambiente Computacional.

Keywords: Geometries, Parametric modeling, *Maple* software, 3D printer.

Resumen

Las Geometrías, Planas, Analíticas y Espaciales, en adelante GEOPAES, como materias de Matemáticas, requieren que los(as) estudiantes sean capaces de visualizar e interpretar figuras geométricas relacionadas con los conocimientos matemáticos a enseñar, previstos en los libros de texto. ¿Cómo ayudar al(a) la Profesor(a) en la enseñanza y al(a) la estudiante en el aprendizaje de los conocimientos GEOPAES, favoreciendo la mencionada visualización e interpretación en la enseñanza donde se utiliza estos libros. Responder a este tipo de preguntas es uno de nuestros objetivos en este artículo, en el que presentamos el modelado paramétrico y la gestión de código de un material didáctico construido en un ambiente computacional, materializado en la impresora 3D, haciendo tangible la “mano libre”. Esta modelización y gestión son prácticas habituales en las investigaciones realizadas en GPEMAC, en el Laboratorio de Visualización Matemática (L@VIM) de la UESC, y han estado al servicio del(de) la Profesor(a) en la enseñanza y del(de) la estudiante en el aprendizaje matemático, así como en otras materias donde GEOPAES encuentra un hábitat en esta institución. Un PCOC impreso en 3D puede contribuir a la identificación y reconocimiento de elementos geométricos notables en el modelo, manipulación y posicionamiento en relación con el sistema de coordenadas, en la aplicación del procedimiento “tomográfico” que examina las intersecciones de superficies que lo delimitan, favoreciendo el análisis de líneas y curvas de nivel, entre otros procedimientos que pueden ser utilizados por el(la) Profesor(a) y el(la) alumno(a). El modelo presentado en este artículo destaca un tipo de estas intersecciones como consecuencia del objeto geométrico observado en los libros de texto.

Palabras clave: Geometrías, Modelado paramétrico, Software de arce, Impresora 3D.

Résumé

Les Géométries Plane, Analytiques et Spatiales, ci-après dénommées GEOPAES, en tant que matières mathématiques, exigent que les étudiants soient capables de visualiser et d'interpréter des figures géométriques liées aux connaissances mathématiques à enseigner, prévues dans les manuels. Comment aider le Professeur dans l'enseignement et l'étudiant dans l'apprentissage des connaissances de GEOPAES, en favorisant la visualisation et l'interprétation susmentionnées dans l'enseignement où ce livre est utilisé ? Répondre à ce type de questions est l'un de nos objectifs dans cet article, dans lequel nous présentons la modélisation paramétrique et la gestion du code d'un matériel pédagogique construit dans un environnement informatique, matérialisé dans l'imprimante 3D, le rendant tangible à "main libre". Cette modélisation et la gestion sont des pratiques courantes dans les recherches menées au GPEMAC, au Laboratoire de Visualisation Mathématique (L@VIM) de l'UESC, et ont servi le Professeur dans l'enseignement et l'étudiant dans l'apprentissage des mathématiques, ainsi que dans d'autres matières où les GEOPAES trouve son habitat dans cette institution. Un PCOC imprimé en 3D peut contribuer à l'identification et à la reconnaissance des éléments géométriques notables dans ce modèle, à la manipulation et au positionnement par rapport au système de coordonnées, dans l'application de la procédure « tomographique » examinant les intersections de surfaces qui le délimitent, favorisant l'analyse des traces et des courbes de niveau, entre autres procédures pouvant être utilisées par l'enseignant et l'étudiant. Le modèle présenté dans cet article met en évidence un type de ces intersections par conséquence de l'objet géométrique observé dans le manuel.

Mots-clés : Géométries, Modélisation paramétrique, Logiciel maple, Imprimante 3D.

Geometric figure in the book and the corresponding freehand PCOC: A code management for the 3D printer

Among the various constituent elements of the social teaching system, the Noosphere, as designated by Chevallard (1992), the textbook, hereinafter referred to as TB, is configured as one of the elements consulted by the Teacher in their effective practice, before entering the classroom, thus contributing to the organization of the objects taught. Furthermore, occupying the second level in the process of didactic transposition, shown later, the TB, as an institutional element (Henriques, Nagamine A, & Nagamine C, 2012, p. 1263), has, over the years, become an object of investigation in education, especially in didactics and mathematics education. This element enables us to reveal the dominant mathematical organizations (DMO) and, consequently, the praxeologies of the objects targeted in the institutions of reference where they are adopted. In this article, we are particularly interested in DMO in the teaching of GEOPAES, based on the hypothesis that plane, analytic, and spatial geometries, as branches or basic disciplines of mathematics, require students to have a visualization and interpretation of geometric figures provided in the TB, whether in a uni, a bi, or a three-dimensional space.

This hypothesis, centered on students' difficulties of visualizing and interpreting geometric objects –especially in three dimensions, where the demands are greater because the phenomenon of "seeing in 3D" translates into difficulty for most students at different levels of education– has already been proven by several researchers in mathematics didactics and mathematics education, interested in the teaching and learning of GEOPAES, among which we can mention: Salazar, Vita, and Almeida (2008), Palles (2012), Bridoux and Nihoul (2015), Marques (2016), and Ramos (2018). These researchers demonstrate that the phenomenon persists over time, across different generations. We maintain, based on our experience and the research reviewed, that student learning also depends on the teaching conditions and (or) the teaching resources that are available to them, as well as on the Teacher. How can we assist Teachers in teaching and students in learning GEOPAES knowing, favoring the visualization and interpretation of geometric figures presented in textbooks used in educational institutions?

Providing at least one answer to this question is one of our objectives in this article, in which we present parametric modeling and code management of a teaching resource, inspired by the analysis of a geometric figure proposed in a TB, produced in the computational environment from the perspective of PCOC (cf. Definition 5 below) and materialized in the 3D printer. With this materialization, a PCOC becomes a concrete material that can be manipulated "freehand", useful for Teachers in teaching, and for students in learning corresponding mathematical concepts in GEOPAES.

To this end, we have organized this article into five parts. In the first part, we present eleven definitions that are important for understanding the article, identified by Definition 1, Definition 2, and so on, along with our quotes or definitions. The second part consists of choosing an institution of reference, a textbook, and a geometric figure as the object of knowing proposed in that book that we analyzed, describing, therefore, the algebraic treatment necessary in the parametric modeling of the PCOC associated with this object, initially using the techniques of the paper pencil environment (cf. Definition 2). In the third part, we present the potential (Henriques, 2021b) of the computational environment that we used in this modeling, namely, the *Maple* software In the fourth part, the modeling and management of the code of the PCOC are implemented (cf. Definition 5) for the 3D printer, using the potential of specific tools of this software, thus establishing an alliance between the paper pencil environment and the computational environment mediating the algebraic treatment carried out in the second part. In the fifth and final part, the PCOC configuration process, also known as slicing, is presented as an intermediate software layer between the model production software and the 3D printer, thereby obtaining the model for 3D printing and providing the final considerations.

Thus, in addition to answering the driving question of this article, the parametric modeling implemented here can serve as a reference for the production of other PCOC as "freehand" manipulable teaching materials helpful in teaching and learning mathematical knowings, using *Maple* or any other software with similar capabilities, thus replicating the modeling and code management mentioned above for the 3D printer.

Some initial definitions

In all sciences, definitions are understood as ways of explaining the concepts or meanings of words, terms, expressions, or even thoughts or ideas that are brought into play. Therefore, we believe it is essential that an article clearly defines the terms used by the authors in the development of the work, providing the reader with a shared understanding of the concepts implemented in the work. In this article, we present the definitions of: didactic transposition; geometric figure; PCOC; geometric sieve; paper_pencil environment; computational environment; visualization; parametric modeling; code management for 3D printing, and concrete material that can be manipulated "freehand," which collaborate in the discourses we employ.

We begin with Duval's (1993, 1995) concepts, highlighting that, contrary to the notion of vision, which provides direct access to a specific object observed with the naked eye, visualization, as defined below, is a more complex and fundamental cognitive process in the development of skills, especially mathematical ones.

Definition 1: Visualization is a cognitive competence based on the production and mental mobilization of a semiotic representation of an object of knowing, which controls the organization of the relationships between significant figural units of this representation, and can be externalized in a given register (our definition).

For example, in the task "describe, in the native language, the surface of the equation given by $mx^2 + ny^2 - pz^2 - q = 0$, being m, n, p and q non-zero real numbers" (Henriques, Farias, & Funato, 2024, p. 19), university students attending Differential and Integral Calculus reveal difficulties in providing the required description, which is manifested by the absence of production and mental mobilization of the semiotic representation corresponding to such equation in the graphic record. They cannot immediately visualize and associate this equation with the reduced form of the one-sheeted hyperboloid equation that they studied in Analytic Geometry.

Referring to the registers of semiotic representation in the context of Duval (1993, 1995, 2012), Henriques and Almouloud (2016) highlight four predominant registers in mathematics, notably: the mother tongue, the algebraic register, the graphic register, and the numerical register, endowed with their signs and rules of conformity that distinguish them from each other.

Definition 1, therefore, contributes to our problem and hypothesis regarding learning in GEOPAES, mobilizing different registers of semiotic representation, and consequently, the didactic transposition. From this perspective, Chevallard (1985) contributed much by disseminating the concept of didactic transposition in the academic community, based on a course taught at the first summer school of the Didactics of Mathematics held in Chambrousse (France), in July 1980. We can translate the idea presented by the author in Definition 2 below.

Definition 2: The didactic transposition is the transformation process that a given object of wise knowing undergoes to be a taught –and consequently learned– knowing (Chevallard, 1985, our translation).

As shown in Figure[6] 1, the process of didactic transposition essentially includes four levels or scales that express the stages and dynamics of a given object of knowing initially thought, created, or discovered by scientists.

[6] We draw attention to the term "Figure" used in academic texts (AT), in view of the notion of geometric figure described in Definition 6. In AT, the term "Figure" numbered sequentially indicates the space reserved in the text in which an object or illustrative scheme is presented that is not necessarily a geometric figure.

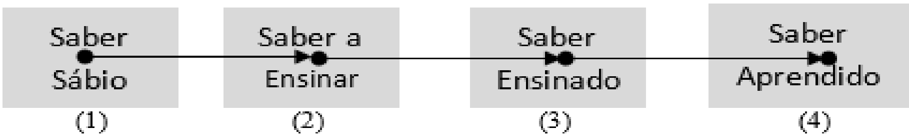


Figure 1.
Didactic Transposition Process (Henriques, 2019, p. 103)

The ***wise knowing*** (level 1) is extra-school or academic knowledge that culturally precedes scientific work. This is the wise knowing of reference, the result of personal discoveries or collective discoveries by groups of people, such as scientists and others. ***Knowing to teach*** (level 2) is described or proposed in *textbooks* and/or in other literary sources, the result of wise knowing. ***Taught knowing*** (level 3) is the one organized by the Teacher based on the knowing to teach. ***Learned knowing*** (level 4) is the knowing acquired by the student from the knowing taught, organized by the Teacher, in a specific institution of reference.

Two terms that emerge in the description of the process of didactic transposition deserve attention: **knowing** and **knowledge**, which seem to be confused with each other. Now, supported by Brousseau (1998), it becomes important to specify the difference between the terms. **Knowing** is a social and cultural environment that enables us to identify, organize, validate, and use knowledge. **Knowledge**, in turn, is the human act or capacity to understand facts, necessary to solve problems that are expected to be mastered by the subject, in particular, students. Some knowledge is not explicable, while others are the knowings taught (level 3) that are converted into environments or strategies for problem solving.

In this article, we pay particular attention to the knowing proposed at the second level as a source of the Teacher, and revealing of dominant mathematical organizations (DMO) in institutions, a knowing that is generally employed in the classroom by both the Teacher and the students, through the mobilization of usual techniques from the paper-pencil and/or computational environments.

Definition 3: A **paper_pencil environment** is a common study space equipped with tools such as paper, pencil, pen, eraser, and others. The board, marker, or chalk also fit into this environment. (Henriques, 2019, p. 26). Articulating practices by applying techniques and tools from the paper_pencil environment with the equivalent work implemented in a computational environment (Definition 4) can contribute positively to the enhancement and learning of the targeted mathematical knowings.

Definition 4: One **computing environment** is the virtual study space made up of tools such as a computer, software, the Internet, a calculator, a 3D printer, etc. (Henriques, 2019, p. 26).

The mobilization of techniques for carrying out GEOPAES tasks in different registers of semiotic representation in both learning environments and the relationship between them when both intertwine in instrumental activities, plays a preponderant role in the parametric modeling of a PCOC (Definition 5), which means that the Teacher must deal very well with the knowing to be taught (level 2) to organize better the knowing taught (level 3), using the techniques of the paper pencil environment. Therefore, this knowledge favors a pertinent modeling of PCOC in a computational environment. But what is a PCOC?

Definition 5: A **project of concrete objects construction** (PCOC) is a descriptive mathematical model designed from the analysis of an institutional task (exercise, example, problem) that results in an object built with concrete materials and (or) with the help of a computational learning environment (Henriques, 2019, p. 32).

Under normal conditions of didactic transposition, a PCOC can be modeled from a geometric figure proposed in the mathematical knowings to teach (level 2), that is, in textbooks or other literary sources, or even elaborated from the Teacher's idealization based on the organization of the knowings taught (level 3). In the scope of geometric constructions, in a paper_pencil environment, the relationships between drawing and geometric figures are classically considered. The drawing, once completed, represents the sensitive world of an object that the mathematician calls a geometric figure and that intervenes in intellectual reasoning. Thus, drawing in mathematics is a representation of a geometric figure, which we define as follows.

Definition 6: A drawing is a tracing on a sheet of paper or a computer screen. When a drawing represents a mathematical object (triangle, parallelogram, for example), it is called a geometric figure**.**

Chaachoua (1997) demonstrates that drawing alternately assumes different roles in teaching. It is considered a physical object in itself, as a representation of a geometric object (geometric figure) or of a physical object.

From this perspective, Parzysz (2002, p. 85) postulates the coexistence of two geometric paradigms of which the Teacher must be aware, to foster his students' learning in geometry:

> A spatial-graphic geometry (G1), in which the objects in play are physical (models, drawings, images on the screen, ...) and the validations of a perceptive nature (point of view, measurement). A proto-axiomatic geometry (G2), in which the objects in play are theoretical and the validations are of a hypothetical-deductive type.

Therefore, drawing can serve as an interface between the facets of geometry, allowing for the prediction of a gradual change in students' practices, from a practical problem where geometry G1 predominates to a geometric problem characterized by a strong presence of geometry G2. (Mathé & Doz, 2019).

"[The] distinction between drawing and geometric figure is generally unnoticed in *Textbooks*, consequently, in traditional teaching" (Henriques, 2001, p. 33). However, the use of concrete manipulable material (Definition 7) "freehand", such as PCOC materialized in the 3D printer, can serve as an incentive to motivate this distinction in the teaching and learning of mathematics. **Definition 7:** A **manipulable concrete material** is any ergonomic and cognitive instrument, tangible to the "free hand," capable of allowing knowledge management, being, therefore, useful in the teaching and learning process of institutional knowledge objects (Henriques, 2019, p. 33).

The construction of the type of material, thus defined, can occur using different methods. Among these methods, we share the construction of geometric objects through parametric modeling, utilizing the geometric-sieve instrumental technique presented in the eighth definition in the context of the registers of semiotic representation (Duval, 1993, 1995), as well as code management for 3D printers.

Definition 8: Geometric sieve is a conservation or choice of part(s) of a curve or surface, necessary in the representation of the corresponding mathematical object in the graphic record (Henriques, Nagamine C, & Serôdio, 2020, p. 258).

The authors explain that "a segment, for example, is part or sieve of a straight line, while discs and polygonal regions are sieves of flat surfaces," among other sieves. If a given surface collaborates with the modeling of a solid, said part or sieve is not necessarily restricted to the boundary of that solid (Henriques, 2021a, p. 135). Hence, the need for a new concept: that of restricted sieve, defined as "the part of a surface that is restricted to the contour of the solid" (Henriques, 2021a, p. 135). Figures 4 and 5 present two illustrations of these concepts:

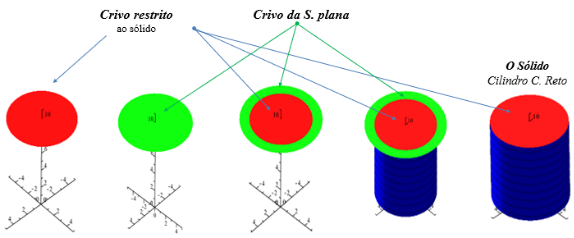


Figure 4.

Illustration of sieves of a flat surface and sieve restricted to the cylinder (Henriques, 2021a, p. 135)

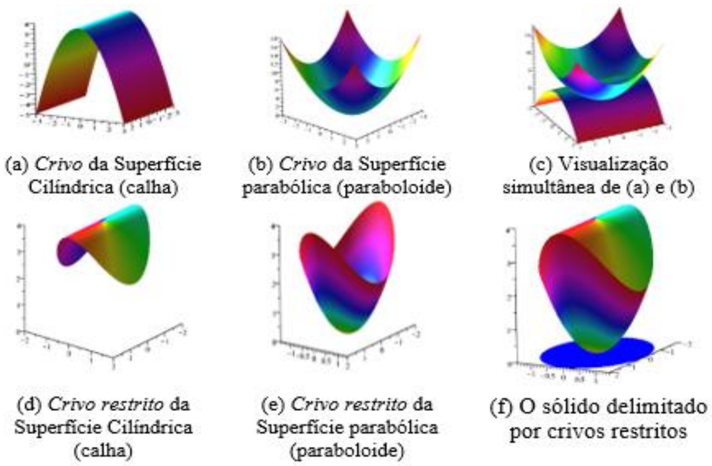


Figure 5.

Illustration of cylindrical surface sieve, paraboloid and sieves restricted to the solid (Henriques, 2021a, p. 135).

The concept of restricted sieve is valid, in an analogous way, in the representation of finite regions, mobilizing, however, the sieves of curves. Polygonal regions, for example, are delimited by restricted sieves of straight lines. It should be understood, therefore, that a PCOC can be formed by a convenient assembly of parts (restricted sieves) of surfaces, constituting the contour of the corresponding solid, as an object or finite geometric figure (cf. Figure 5(f)).

The application of the geometric sieve technique contributes to the production of teaching materials that can be manipulated "freehand," from the PCOC perspective, in which we use parametric modeling and the construction by coordinate system (CCS) technique, defined below. It is worth highlighting, first of all, to situate the reader and avoid conflicts of thought from the perspective of mathematics education, in which the term modeling is consolidated in a line or field of investigation, that in the research carried out by our team, involving the production of the materials mentioned before, we use parametric modeling (Definition 9).

Definition 9: Parametric modeling is a method of generating and manipulating geometric objects, such as curves and surfaces in a computational environment, and is based on the connection of these objects and their interrelations mediated by specified parameters that can be automatically changed by the environment or by the subject in real time, without loss of the targeted geometry (Henriques, 2021, p. 107).

In the context of this modeling, there are numerous possibilities for manipulating the geometric objects above based on the associated algebraic objects, expressed in various coordinate systems. This thought leads to the applicability of the technique presented in the following definition, known as construction using a coordinate system.

Definition 10: Construction using a coordinate system (CCS) is a technique for representing curves and surfaces of known equations or graphs of functions in a computational environment based on the specification of a two-dimensional (2D) or three-dimensional (3D) coordinate system using corresponding commands (our definition).

The materialization of a PCOC produced in a computational environment for the 3D printer, using the techniques defined until then, so that it is tangible freehand in the teaching and learning of inherent mathematics, involves the management of a code.

Definition 11: Code management of a model is the development or use of a process to convert the model into a file with an extension recognized by the 3D printer, either directly or indirectly, through another model configuration environment that communicates with this 3D printer (Henriques, 2021, p. 117).

Considering the theoretical elements presented in this part of the article, we hope that the subsequent work, which references these elements, will be plausible to the reader, especially to the Teacher who may be interested in producing teaching materials from this perspective, and to the student in mathematical learning. Indeed, it is essential to situate teaching, thus choosing an institution of reference. For this article, we chose, in addition to the institution, a textbook and a geometric figure referring to an object of knowledge proposed in that book. These choices favor the analyses and mathematical treatment necessary in the parametric modeling of the PCOC associated with the geometric figure, mobilizing techniques from the papel_pencil environment (cf. Definition 3).

Choice of the institution of reference

We chose exact and technological science courses as the institution of reference and the book *Geometria Analítica* (*GA*)*: Um tratamento vetorial* [Analytic Geometry: A vector treatment] by Camargo and Boulos (2005) as an institutional element, where we highlight the teaching of quadrics, especially the hyperboloid of an elliptic sheet, and surface intersection curves as a mathematical object of reference for PCOC modeling. The choice of this object is justified by our understanding that it is one of the mathematical objects equipped with geometric figures that students who enroll in Analytic Geometry and Differential and Integral Calculus have difficulty in constructing using the techniques of the paper_pencil environment, and are, therefore, the target audience for this production. In the theory presented in the 3rd edition, page

407, the authors propose the following definition:

> A Quadric $\mathbf{\Omega}$ is **a hyperboloid of one sheet** if there are positive real numbers *a, b, c,* and an orthonormal coordinate system regarding which $\mathbf{\Omega}$ can be described by the equation called the reduced equation of $\mathbf{\Omega}$ (Camargo & Boulos, 2005, p. 407).

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad \text{(Eq1)}$$

The hyperbolic surface of one sheet (Eq1) intersects the axis_*x* and axis_*y* at± points (*a*, 0, 0) and (0, ±*b*, 0) of that system, respectively, being *a, b,* and *c* didactic variables that assume non-zero real values or numbers. Whereas the intersection of the hyperboloid with the axis_*z*, is an empty set, since $z^2=-c^2$ is insoluble in the set of real numbers. On page 409, the authors present, in graphic form, the intersection of $\mathbf{\Omega}$ with equation planes *y=k,* parallel to plane_*xz*, in cases where $0 < k < b$, $k = 0$, and $k >$ b, as shown in Figure 6. However, at no point do they teach how the sieves of the surfaces of an elliptic hyperboloid of one sheet and the planes presented are constructed. The lack of explanation or instruction on production techniques for these sieves may also contribute to why students struggle to learn the corresponding objects.

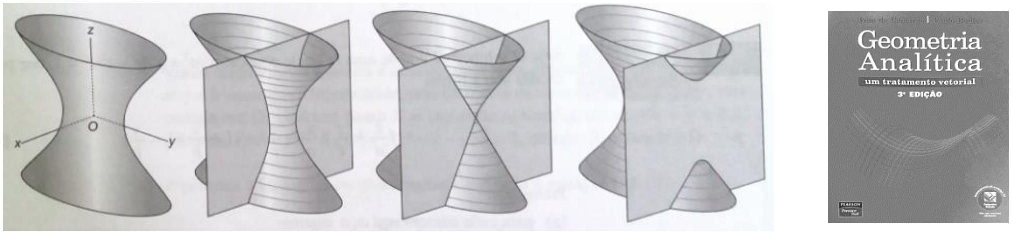


Figure 6

Hyperboloid of one sheet and planes parallel to the axis_z (Camargo & Boulos, 2005, p. 409)

Thus, in addition to answering the question presented previously, our main objective in this article is to establish a relationship between a geometric figure presented in a textbook and a corresponding PCOC model produced in a computational environment, materialized in the 3D printer, therefore tangible in free hand, observable in all possible directions, and valuable in research and teaching. Hence the title of this article.

Now, there are different ways to model the chosen object. To use the potential of the tools presented, further on in Figure 8, and the theoretical elements considered in the previous section, we chose to rewrite (Eq1) as shown in (Eq2), thus considering the elliptical hyperboloid of one sheet, whose traces by planes perpendicular to the axis_z and the corresponding level curves are ellipses. We then establish parameter declarations applicable in parametric modeling, constraining the mentioned hyperboloid to a metric-specific PCOC model.

$$z=\pm\sqrt{\left(\frac{c}{a}x\right)^2+\left(\frac{c}{b}y\right)^2-c^2} \qquad \text{(Eq2)}$$

Considering the equations $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = z$ that allow establishing the relationship between Cartesian coordinates (x, *and*, z) and cylindrical coordinates (r, θ, z) from a point **P** of three-dimensional space, being r and θ the polar coordinates of point P' (orthogonal projection of the point **P** on plane_*xy*), we can, from (Eq2), obtain a parameterization of the surface **Ω**, denoted by [X(u,v), Y(u,v), Z(u,v)] with $u, v \in \mathbb{R}$, in syntaxes of *Maple*, and represented in Henriques (2021, p. 96) by:

$$S(u,v) = \begin{cases} x = \mathrm{X}(u,v) \\ y = \mathrm{Y}(u,v), u \in I \subseteq \mathbb{R}, v \in J \subseteq \mathbb{R} \\ \mathrm{z} = Z(u,v) \end{cases}$$

in which $\mathrm{X}(u,v), \mathrm{Y}(u,v), Z(u,v)$ indicate the parametric equations of a surface S. Thus, surface **Ω** admits the parameterization indicated in (Eq3), making $u = z$ and $v = t$:

$$\begin{cases} \mathrm{X}(z,t) = \frac{a}{c}\sqrt{z^2+c^2}cos(t) \\ \mathrm{Y}(z,t) = \frac{b}{c}\sqrt{z^2+c^2}sen(t), z \in \mathbb{R}, t \in [0, 2\pi] \\ \quad Z(z,t) = z \end{cases} \qquad \text{(Eq3)}$$

From this parameterization achieved by performing algebraic treatment of Eq2 with paper_pencil environment techniques, expressing the polar radius r in function of mz, we can move on to the work required to implement the PCOC model **Ω** in the computing environment. To achieve this, it is necessary to understand the potential of the tools in the environment used, from the perspective of Rabardel's instrumental approach, as presented in Henriques (2021), which is essential for such implementation.

Potentialities of the *Maple* computing environment for modeling a PCOC

In the generation and manipulation of geometric objects mentioned in the definition of parametric modeling, we consider the paper_pencil environment and the *Maple* computational environment. *Maple* was developed by a group of researchers from the University of Waterloo, Canada, dating back to the 1990s, designed for advanced mathematics to provide tools for the study of various mathematical objects recognized at all school levels.

To produce a PCOC model in this software, it is necessary to mobilize some tools or commands that have the potential to represent mathematical objects in the graphic record. In this mobilization, the techniques presented above play a fundamental role in the construction process, according to their requirements in the preliminary mathematical treatment, which must be carried out by the subject in the algebraic register, in favor of parametric modeling, in the context defined in this article, aiming at mathematical learning. This treatment helps in the convenient use of software commands.

It is worth noting that in *Maple*, commands are organized into packages, which can be understood as a way of organizing specific areas of mathematics within the software. Figure 7 illustrates the packages we utilize during parametric modeling and PCOC code management for 3D printers to facilitate the visualization of objects in the graphic register from symbolic structures or instructions implemented in command lines or prompts.

Table 1.

Packages that contain commands for viewing objects in the graphic register (Farias, Funato, & Cattai, 2016, p. 8)

1	The package named "*plots*" accessed according to the syntax *with(plots)*;
2	The package named "*plottools*" accessed according to the syntax *with(plottools)*;
3	The package named "*FileTools*" accessed according to the syntax *with(FileTools)*;

Thus, to promote the effective functioning of the instructions implemented in the *Maple*, these packages must be presented correctly with their respective syntaxes. Executed by pressing the "*Enter*" key or by clicking on the !!! icon in the tools menu, *Maple* returns the set of commands available in each. From there, the subject must develop skills regarding the desired modeling, providing the *Maple* instructions from the prompt via the keyboard. In the parametric modeling that we present, we use the commands and their respective syntaxes shown in Figure 8, without describing their potential, as this is not the purpose of this article. However, such descriptions can be found in Farias, Funato, and Cattai (2016) and Henriques (2021).

Table 2.

Commands used in PCOC modeling and their respective syntaxes (Farias, Funato, & Cattai, 2016, p. 8).

1	*plot3d([expreu, exprev, exprew], s=a..b, t=c..d, opts)*
2	*display(objects, opts)*
3	*JoinPath(components, opts)*
4	*exportplot(fname, p, opts)*
5	*importplot(fname, opts)*

The first two syntaxes collaborate with parametric modeling, properly speaking, while the last three intervene in the management of the modeled PCOC code, through the use of the first two syntaxes. Based on this knowledge, we present the desired modeling below.

Parametric modeling of the PCOC of Ω (elliptical hyperboloid of one sheet) sectioned by a plane parallel to the axis_z

To provide better monitoring of the implementation of the instructions in the *Maple*, we organize the modeling and code management of the PCOC of **Ω** in instruction execution groups. To generate and manipulate geometric objects corresponding to the desired PCOC, it is necessary first to determine the desired dimensions for the model. In this modeling, we set the following measures for the PCOC of **Ω**:

- ❖ 12 cm in height.
- ❖ 4 cm on axis_*x* of the intersection curve with the plane of symmetry in relation to its height.
- ❖ 7 cm on axis_*y* of the intersection curve with the plane of symmetry in relation to its height.

For convenience, we choose the plane of symmetry to be the plane_*xy*. We load the necessary packages, as shown in Figure 7, ending each instruction with a colon to load the package without displaying the specific set of commands. Next, we enter the global variables, thus forming the first group (GE1).

First execution group: Loading required packages and declaring global template variables	**(GE1)**
with(*plots*): *with*(*plottools*), *with*(*FileTools*):	
Factor:=10:	# the metrics of geometric objects visualized in *Maple* are given in *mm*. The *Factor* variable has the role of transforming *mm* to *cm*.
a:=2: *b:*=3.5: c:=2.7	# values of the coefficients or teaching variables of the hyperboloid for the PCOC of **Ω**.
z1:=-6: *z2:*=-*z1:*	# equation of the lower plane and the upper plane that delimit the height of the PCOC of **Ω**..
zmin ≔ *z*1	# minimum value assumed by z in PCOC of **Ω**..
zmax ≔ *z*2	# maximum value assumed by z in PCOC of **Ω**..
dr:=0.45*:*	# increment of values.

To produce the PCOC of **Ω** by applying the techniques defined previously, we can mobilize the idea of superimposing the traces of this PCOC, which are ellipses along the eixo_z. So, doing $x = \frac{a}{c} rcos(t)$ and $y = \frac{b}{c} rsen(t)$ and substituting it into the equation indicated in Eq2, we have the following result:

$$z = \sqrt{r^2 - c^2} \text{or } z = -\sqrt{r^2 - c^2} \qquad \text{(Eq4)}$$

Isolating *r* in one of these equations, we implement the second group (GE2) where we define, using the command *unapply*, the functions corresponding to the parametric equations presented previously in Eq3, which model the PCOC of **Ω**, considering the chosen measurements, since every model to be printed in 3D requires the specification of dimensions based on the mathematical model of the object under study. In this case, the model by Camargo and Boulos (2005, p. 407), shown in Eq1.

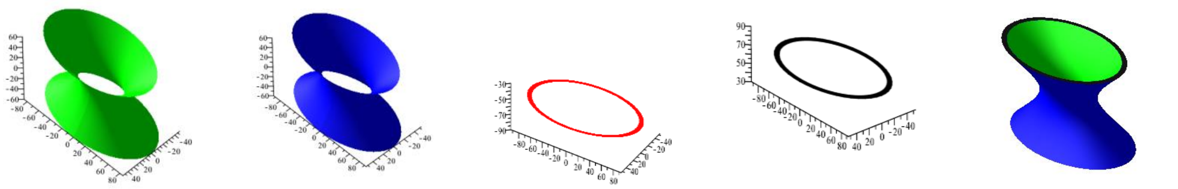
It is worth highlighting that the 3D printer we have does not print open surfaces, but rather closed ones delimiting a three-dimensional (solid) space. Unlike the ellipsoid, for example, which is a closed surface delimiting the elliptical sphere, which is a solid, the hyperboloid of one sheet is an open surface. Thus, the models we produce are implemented as a shell with thickness dr, obtaining a closed surface. Hence, the repetition of parametric equations $x = \mathrm{X}(u, v)$ and $y = \mathrm{Y}(u, v)$, in the second group, using the increment dr in each repetition.

Second execution group: Definition of functions corresponding to the parametric equations of the PCOC of Ω	(GE2)
Implemented instruction	**Instruction description**
$Xi(z,t) = unapply\left(\frac{a}{c}\sqrt{z^2+c^2}.\cos(t), (z,t)\right)$	# Component X of the parameterization of the hyperboloid of one sheet, internal.
$Yi(z,t) = unapply\left(\frac{b}{c}\sqrt{z^2+c^2}.\cos(t), (z,t)\right)$	# Component Y of the parameterization of the hyperboloid of one sheet, internal.
$Xe(z,t) = unapply\left(\frac{a}{c}\left(\sqrt{z^2+c^2}+dr\right).\cos(t), (z,t)\right)$	# Component X of the parameterization of the hyperboloid of one sheet, external.
$Ye(z,t) = unapply\left(\frac{b}{c}\left(\sqrt{z^2+c^2}+dr\right).\cos(t), (z,t)\right)$	# Component Y of the parameterization of the hyperboloid of one sheet, external.
$Z(z,t) = unapply(z, (z,t))$	# Component Z of the parameterization of the hyperboloid of one sheet.

Then we implemented, in the third modeling execution group, the sieve of **Ω** (Elliptical Hyperboloid of One Sheet), hereinafter also represented by EHOS, in shell form, using these definitions and the global variables. It is, therefore, the generation and manipulation of geometric objects, in their connections and their interrelations, mediated by previously declared parameters or variables. To do this, we consider four local variables that store the sieves, identified by: *HipI* (variable that stores the internal EHOS sieve), *HipE* (variable that stores the external sieve of EHOS), *AnelInfer*, and *AnelSuper* (variables that store the ring of the lower cover and the upper cover of the shell, respectively). With these choices, we enter the command lines (or traces) of *Maple* with the instructions presented in GE3 using the syntax ***plot3d([X(u,v), Y(u,v), Z(u,v)], u=umin..umax, v=vmin..vmax, <opções>)*** of *Maple* command *plot3d* (Henriques, 2021a, p. 95). The seventh instruction presented in this group gathers the results stored in the four variables in the same coordinate system, assigning this result to the variable "OmodeloPCOC."

Third group of execution: modeling of the cylindrical shell of a hyperboloid of one sheet	(GE3)
raiome:= Xi(zmax,α): # radius smaller than the shell ring. *raioMa:= zmax : # radius bigger than the shell ring.* *HipI:= plot3d(Fator*[Xi(z, t), Yi(z, t), Z(z, t)], z = z1 .. z2, t = alpha .. beta, color = green)* *HipE:= plot3d(Fator*[Xe(z, t), Ye(z, t), Z(z, t)], z = z1 .. z2, t = alpha .. beta, color = blue)* *AnelInfer:= plot3d(Fator*[Xe(z, t), Ye(z, t), zmin], z = raiome .. raioMa, t = alpha .. beta, color = red)* *AnelSuper:= plot3d(Fator*[Xe(z, t), Ye(z, t), zmax], z = raiome .. raioMa, t = alpha .. beta, color = black) :* *OmodeloPCOC:= display(HipI, HipE, AnelInfer, AnelSuper, labels = [x, y, z]) :* *display(OmodeloPCOC, scaling = constrained)*	

After executing each instruction with the "*Enter,*" *Maple* returns the expected result, as presented in Figure 7.



(a) HipI inner sieve (b) HipE outer sieve (c) Lower Ring (d) Upper Ring (e) The PCOC model

Figure 7.

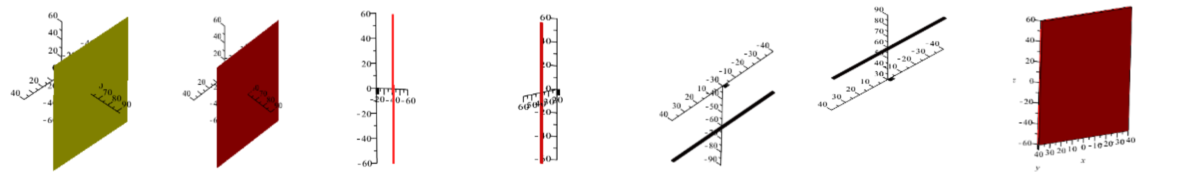
Visualization of the result of the implementation of the cylindrical shell of an elliptical hyperboloid of one sheet (Research data)

A sieve of a plane ⊥ orthogonal to the plane_*xy* must then be modeled in the subsequent execution group (GE4). When relating this PCOC model to the corresponding geometric figure in the book (Camargo & Boulos, 2005, p. 409), we modeled the plane so that it is parallel to plane_*xz*, of equation $y = k$. Considering the options indicated by the authors of the book, we chose the case in which $k > b$. Thus, knowing, through the choice made previously in which $b = 3.5$, we can consider any value greater than 3.5 and smaller than the radius of the longest trace (ellipse) of the PCOC. So, doing $k = 6$ we fix the flat surface of equation $y = 6$. It is worth remembering, however, as highlighted above, that this plane must also be modeled with thickness. This requirement leads to the modeling of a sufficiently thin parallelepiped with a height equal to that of the PCOC.

Fourth execution group: parametric modeling of a "thick (shell)" plane	(GE4)

```
k:= 6: drk:= 0.2:# increment for y.
xmax:=2*a: xmin:=- xmax: ymax:=2*a: ymin:=- ymax:
Planomiy:= plot3d(Fator*[u, k, v], u = xmin .. xmax, v = zmin .. zmax, color = yellow):
Planomay:= plot3d(Fator*[u, k+drk, v], u = xmin .. xmax, v = zmin .. zmax, color = red):
Planomix:= plot3d(Fator*[xmin, u, v], u = k .. k+drk, v = zmin .. zmax, color = red):
Planomax:= plot3d(Fator*[xmax, u, v], u = k .. k+drk, v = zmin .. zmax, color = red):
Planomiz:= plot3d(Fator*[u, v, zmin], u = xmin .. xmax, v = k .. k+drk, color = black):
Planomaz:= plot3d(Fator*[u, v, zmax], u = xmin .. xmax, v = k .. k+drk, color = black):
Planokasca:= display(Planomiy, Planomay, Planomix, Planomax, Planomiz, Planomaz, labels = [x, y, z]):
display(Planokasca, scaling = constrained);
```

After each instruction executed with the "*Enter*" command, *Maple* returns the expected result, that we present in Figure 8.



(a) Sieves of planes of equations $y = 6$and $y = 6 + drk$

(b) Sieves of planes of equations $x = xmin$and $x = xmax$

(a) Sieves of planes of equations $z = zmin$and $z = zmax$

(d) Thin parallelepiped (Planokasca)

Figure 8.

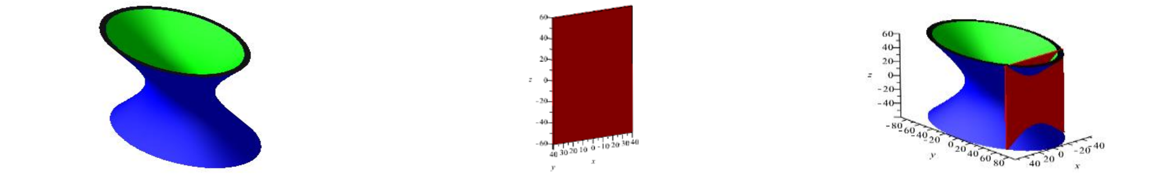
Visualization of the result of implementing a parallelepiped shell (Research data)

As mentioned previously, the colon placed at the end of each instruction allows one to hide the result of each instruction given to *Maple*, which can be recovered whenever necessary. Using the tool or command display, you can view the results stored in the variables Planokasca and OmodeloPCOC together, as occurs with the implementation presented in the fifth execution group.

Fifth group of execution: of the plane "with thickness (shell)" and of the shell of the hyperboloid	(GE5)

```
display(OmodeloPCOC, Planokasca, scaling = constrained, labels = [x, y, z])
```

The execution of the instruction presented in this group with the "*Enter*" key returns the result shown in Figure 9(c), while 9(a) and 9(b) display the results obtained in GE3 and GE4, respectively.



(a) EHOS sieve in shell or the PCOC model | (b) Thin parallelepiped or Sieve of the shell plane | (c) The PCOC model sectioned by the sieve of the shell plane

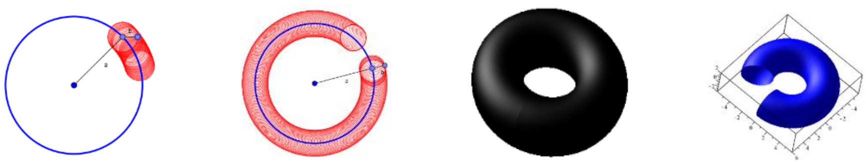
Figure 9.

Visualization of the PCOC EHOS sectioned by a parallelepiped shell (Research data)

It is noteworthy, as expected, that the intersection or equivalently a trace of the EHOS generated by a plane parallel to the elliptical axis of revolution of this surface is a hyperbola contained both in the EHOS and in the referred plane, as can be seen in Figure 9(c), according to the figure presented in the textbook of reference of this PCOC.

Furthermore, the traces or intersections of planes perpendicular to the said axis are elliptic curves. The construction, that is, the modeling of those lines in the computational environment *Maple,* aiming at 3D printing, also requires an implementation with thickness. This environment features the "*thickness*" tool, which has the potential to assign thickness to geometric objects. However, the result obtained using this tool is merely visual on the computer screen; the 3D printer does not identify it, as the set of points that compose the intersection curve belongs to the surfaces that determine it. To solve this problem, we use the concept of a torus. According to Henriques (2021, 98),

> In mathematics, the torus is a surface that consists of a topological space homeomorphic to the product of two circles, each contained in two mutually orthogonal planes [cf. Figure 10(c) and (d)], and is shaped like a tire inner tube of a vehicle.



(a) (b) (c) (d)

Figure 10.

Illustration of a geometric process for generating a torus (Henriques, 2021a, p. 98)

The author adds that:

> In the geometric context, it can therefore be said that a torus, as a surface in the graphic record, consists of the three-dimensional geometric locus formed by the revolution of the smaller circle of radius ***b*** (*Generatrix*) around the larger circle of radius ***a*** (*Directrix*) (cf. Figure 10 (a) and (b)). In the algebraic record, the surface of a **torus** is parameterized with the following equations:

$$\begin{cases} X(u,v) = (a + b\cos(v))\cos(u) \\ Y(u,v) = (a + b\cos(v))\sin(u) \;, \quad u,v \in [0,2\pi], \text{sendo } b < a \\ Z(u,v) = b\sin(v) \end{cases}$$

in which ***a*** is the radius of the ***directrix*** *circumference* and ***b*** is the radius of the ***generatrix*** *circumference* of the torus (Henriques, 2021a, p. 98).

Based on this knowledge, we implement the elliptical lines of the PCOC model of **Ω**, extracting from the parameterization implemented in GE3 the radius of the generatrix and the directrix of each torus, thus favoring the declaration of local variables presented in the sixth execution group. Attention: Do not confuse the didactic variables a and b present in the theoretical model of the torus above with the didactic variables that have the same identity in the model **Ω**.

Declaration of local variables of the parametric modeling of tori on the PCOC of **Ω**	**(GE6)**
Implemented instruction	**Instruction description**
re:= 0.2	*# Value of the radius of the **generatrix** circumference of the torus in the elliptical trace of the EHOS*
$raz0 := \frac{a}{c}\cdot \text{sqrt}(0^2 + c^2) + dr : rbz0 := \frac{b}{c}\cdot \text{sqrt}(0^2 + c^2) + dr :$	*# Radius of the **directrix** elliptic torus, where raz0 is the coefficient of x and rbz0 is the coefficient of y in the ellipse on the plane of equation z=0*
$raz1p5 := \frac{a}{c}\cdot \text{sqrt}(1.5^2 + c^2) + dr : rbz1p5 := \frac{b}{c}\cdot \text{sqrt}(1.5^2 + c^2) + dr :$	*# Radius of the **directrix** elliptic torus, where raz1p5 is the coefficient of x and rbz1p5 is the coefficient of y in the ellipse on the plane of equation z=1.5*
$raz3 := \frac{a}{c}\cdot \text{sqrt}(3^2 + c^2) + dr : rbz3 := \frac{b}{c}\cdot \text{sqrt}(3^2 + c^2) + dr :$	*# Radius of the **directrix** elliptic torus, where raz3 is the coefficient of x and rbz3 is the coefficient of y in the ellipse on the plane of equation z=3*
$raz4p5 := \frac{a}{c}\cdot \text{sqrt}(4.5^2 + c^2) + dr : rbz4p5 := \frac{b}{c}\cdot \text{sqrt}(4.5^2 + c^2) + dr :$	*# Radius of the **directrix** elliptic torus, where raz4p5 is the coefficient of x and rbz4p5 is the coefficient of y in the ellipse on the plane of equation z=4.5*
$raz6 := \frac{a}{c}\cdot \text{sqrt}(6^2 + c^2) + dr : rbz6 := \frac{b}{c}\cdot \text{sqrt}(6^2 + c^2) + dr :$	*# Radius of the **directrix** elliptic torus, where raz6 is the coefficient of x and rbz6 is the coefficient of y in the ellipse on the plane of equation z=6*

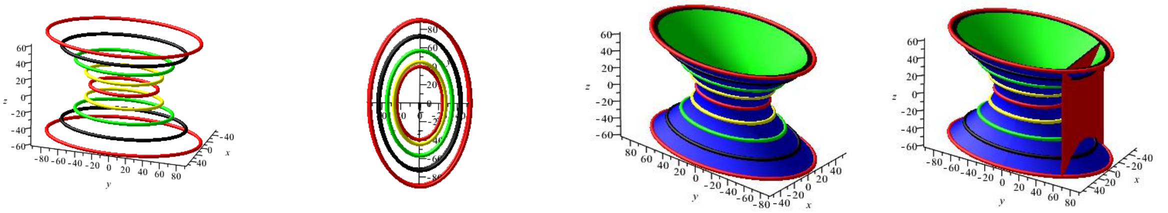
We can observe, from GE6, that the directrix radii of the tori are related to the parameterization of the outer surface of the shell of the model of the PCOC of **Ω,** so that the lines are highlighted in the PCOC externally. Considering the declaration of variables presented in GE6, the respective tori are implemented in the seventh execution group, representing the lines or intersections in question expressing reliefs in the PCOC of **Ω**.

Modeling of traces or lines (intersections of planes orthogonal to the axis_z of the hyperboloid) as tori	**(GE7)**

```
Tracoz0:= plot3d(Fator*[(raz0+re*cos(v))*cos(t), (rbz0+re*cos(v))*sin(t), re*sin(v)+0], v = alpha .. beta, t = alpha .. beta, color = red):
Tracoz1p5p:= plot3d(Fator*[(raz1p5+re*cos(v))*cos(t), (rbz1p5+re*cos(v))*sin(t), re*sin(v)+1.5], v = alpha .. beta, t = alpha .. beta, color = yellow):
Tracoz1p5n:= plot3d(Fator*[(raz1p5+re*cos(v))*cos(t), (rbz1p5+re*cos(v))*sin(t), re*sin(v)-1.5], v = alpha .. beta, t = alpha .. beta, color = yellow):
Tracoz3p:= plot3d(Fator*[(raz3+re*cos(v))*cos(t), (rbz3+re*cos(v))*sin(t), re*sin(v)+3], v = alpha .. beta, t = alpha .. beta, color = green):
Tracoz3n:= plot3d(Fator*[(raz3+re*cos(v))*cos(t), (rbz3+re*cos(v))*sin(t), re*sin(v)-3], v = alpha .. beta, t = alpha .. beta, color = green):
Tracoz4p5p:= plot3d(Fator*[(raz4p5+re*cos(v))*cos(t), (rbz4p5+re*cos(v))*sin(t), re*sin(v)+4.5], v = alpha .. beta, t = alpha .. beta, color = black):
Tracoz4p5n := plot3d(Fator*[(raz4p5+re*cos(v))*cos(t), (rbz4p5+re*cos(v))*sin(t), re*sin(v)-4.5], v = alpha .. beta, t = alpha .. beta, color = black):
Tracoz6p := plot3d(Fator*[(raz6+re*cos(v))*cos(t), (rbz6+re*cos(v))*sin(t), re*sin(v)+6], v = alpha .. beta, t = alpha .. beta, color = red):
Tracoz6n := plot3d(Fator*[(raz6+re*cos(v))*cos(t), (rbz6+re*cos(v))*sin(t), re*sin(v)-6], v = alpha .. beta, t = alpha .. beta, color = red):
```

```
display(Tracoz0, Tracoz1p5p, Tracoz1p5n, Tracoz3p, Tracoz3n, Tracoz4p5p, Tracoz4p5n, Tracoz6p, Tracoz6n)
Tracos := display(Tracoz0, Tracoz1p5p, Tracoz1p5n, Tracoz3p, Tracoz3n, Tracoz4p5p, Tracoz4p5n, Tracoz6p, Tracoz6n, OmodeloPCOC)
display(OmodeloPCOC, Planokasca, Tracos, scaling = constrained, labels = [x, y, z])
```

The execution of these instructions provided to *Maple* in this parametric modeling returns the results presented in Figure 11, with Figure 11(b) being obtained by direct manipulation of Figure 11(a) with the mouse, thus projecting this Figure 11(a) onto plane_*xy*.



(a) Torus traces | (b) Level curves | (c) Torus traces over the PCOC model of **Ω** | (d) Features of the PCOC model of **Ω** and plane with "thickness"

Figure 11.

Visualization of PCOC lines of **Ω** *and the level curves corresponding to the lines in the xy_plane (Research data)*

With this result, the parametric modeling of the PCOC of **Ω** is concluded, and we continue production by managing the code of this model with a view to its 3D printing, thus implementing the eighth execution group that must contain the following data:

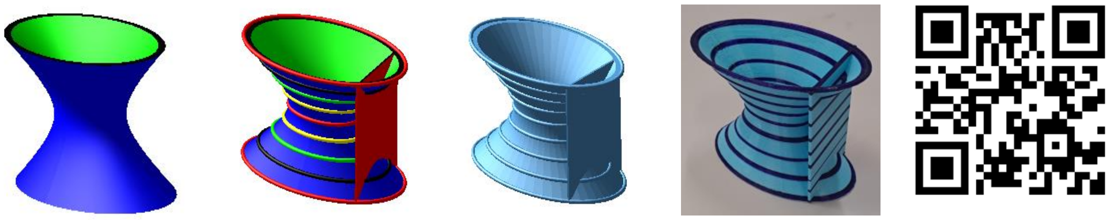
- The last instruction presented in GE7 assigned to a variable;
- A path managed by the command *JoinPath* of the package *FileTools* presented in Figure 7;
- An instruction of the *exportplot* command that recognizes the path to export the code of the PCOC model of the elliptic hyperboloid of one sheet (EHOS) in STL to a documents folder on the platform being used (*Linix*, *Macintosh*, *Windows* etc.)
- An instruction of the *importplot* command that rescues the model in ***STL*** to be viewed in the PCOC model computational production environment (Optional).

In the parametric modeling of the PCOC of **Ω** (the EHOS) developed in this article uses the Windows platform. Therefore, we define a folder in Windows that serves as a storage space for the model code in *STL*. This folder is not necessarily the same one that contains the file *Maple* of the modeling implemented in execution groups. Thus, the eighth execution group was drawn up with the following instructions:

Management of the code of the elliptic hyperboloid of one sheet **(GE8)**

```
HiperbUmaFolha := display(OmodeloPCOC, Planokasca, Tracos, scaling = constrained, labels = [x, y, z]);
Caminhostl := FileTools:-JoinPath(["C: ", "ModelosMaple", "HiperboloideUmaFolha.stl"], platform = "windows");
exportplot(Caminhostl, HiperbUmaFolha);
importplot(Caminhostl);
```

Executing these instructions with the "*Enter*" key, *Maple* returns the result shown in Figure 12(c). It is essential to note that this management is applicable to the creation of STL codes for any PCOC model implemented in Maple, by adapting the variables used.



(c) The PCOC model of Ω in shell | (b) Model with traction and sieve of a plane | (c) PCOC model of Ω in STL | (d) PCOC Model of Ω printed in 3D[7] | QR Code of the video of the printing 3D of this PCOC

Picture 12.

View of the PCOC model of $\boldsymbol{\Omega}$ *(elliptic hyperboloid of one sheet) in the version* Maple*, in* STL, *and the 3D printed version, being accessible freehand*

Figure 12(d) shows the expected 3D printing result. Thus, the primary objective of this article is to share with the academic and scientific community our contribution regarding the possibilities of producing 3D printed models or teaching materials, highlighting their potential and relationships with the geometric figures available in textbooks, thereby facilitating their "freehand" handling. In this way, we hope to have participated effectively in institutional interests concerning the development of teaching materials that can be materialized in the 3D printer for educational purposes, both in teaching in mathematics classrooms and research and/or university extension.

From this perspective, the printed model can be explored in different ways. In the research conducted at GPEMAC, for each model that makes up the L@VIM collection, including this one, an experimental device is developed, organized with at least one task generator (TG) (Henriques & Almouloud, 2022, p. 379), managing a significant number of tasks from the 3D printed model. Each device constitutes a didactic sequence (DS) within the scope of Artigue's (1988) didactic engineering, which is proposed to students. Table 3 presents the device organized based on the PCOC of **Ω.**

Table 3:

Experimental device around the PCOC model of $\boldsymbol{\Omega}$ *3D printed (Research data)*

GPEMAC / Visualização L@VIM Matemática	SANTA CRUZ STATE UNIVERSITY - UESC DEPARTMENT OF EXACT SCIENCES (DCEX) Research Group on Teaching and Learning Mathematics in a Computational Environment Laboratory for Mathematical Visualization (L@VIM)	IN ALTUM
EXPERIMENTAL DEVICE		
Experimental device for analyzing effective student practices in the study of plane, analytic, and spatial		

[7] https://youtu.be/m68funXkm2U (the link to access the printing video for this model) and the *STL* code will be made available here for download.

geometries (**GEOPAES**) using PCOC models[8] in higher education.		
Class professor (optional):		Date:
Student name (optional):		Course:
DS Unitary Session		
Task Generator (GT_1)		
Carefully observe the PCOC model in your hands, which was obtained through materialization on the 3D printer. If necessary, use an appropriate measuring instrument to perform each task (t_n) of the GT_1, with $n = 1, 2, \ldots, 12$, explaining each stage of implementation in the native or natural language.		
t_1	Represent the PCOC model that is in your hands, in a three-dimensional coordinate system, on the answer sheet, with any scale, respecting the measurements of the model elements.	
t_2	Provide the name given in analytic geometry to the PCOC model that you just represented on the answer sheet.	
t_3	Gently run your hands over the model, feeling the curves in relief on its contour, and give the name attributed to these curves in Differential and Integral Calculus.	
t_4	Describe in the native language register how the named curves are obtained when performing the t_3.	
t_5	Provide the name assigned to these curves projected orthogonally onto the model's support plane.	
t_6	Represent, in the algebraic record and in the graphic record, the curves named in the realization of the t_5.	
t_7	Provide the measurement of the minor axis and major axis of the smallest curve and the largest curve with notable relief in this PCOC.	
t_8	Provide the general equation, in algebraic form, of the mathematical model associated with this PCOC.	
t_9	Provide the specific equation of this PCOC that is in your hands, observing the result obtained when performing the t_1.	
t_{10}	Describe the curve generated by the intersection of the sieve of plane (CP) modeled with thickness with the PCOC.	
t_{11}	Compare the result obtained when performing the t_1, and indicate the coordinate plane with which the CP described in t_9 is parallel.	
t_{12}	Represent the curve described in the realization of t_{10}, in the algebraic register.	

Each GT_1 task then goes through a priori analysis from the perspective of a DS, revealing the objective, the mathematical and didactic analysis, the didactic variables, the prerequisites and competencies, and the expected results for each task. Once the a priori analysis is completed, this device must enter the application, a posteriori analysis, and validation stages of a DS, which are not included in this article and are indicated for future work. We highlight that the twelve tasks managed in the GT_1 do not exhaust the management of tasks for this PCOC that can be worked on with students in the paper pencil environment across different institutions. In addition, other task generators can be created with this PCOC in hand, allowing students to model them using the computational environment *Maple* or other environments, such as *GeoGebra* software, and manage the corresponding code for the 3D printer.

For 3D printer technology to be truly useful in education we must emphasize that its use is not as complex as one might imagine, as it works by analogy with the usual practice of 2D printing on legal paper, where the user must initially have a document that has been previously prepared by someone or by the users themselves. Therefore, parametric modeling and code management for 3D printing are necessary practices in mathematics education since, besides favoring "freehand" access to objects, they can contribute to the development of students'

[8] Concrete object construction project (Projeto de Construção de Objetos Concreto - PCOC).

inherent mathematical competencies, such as: analyzing each instruction implemented in the computational environment and extracting the corresponding algebraic representations; analyzing the printed model and identifying inherent mathematical concepts or objects; analyzing the intersections (traces), the level curves, and identifying the corresponding algebraic representations, as proposed in the experimental device shown in Table 3.

In this way, we hope that the concept of geometric figures in the book and the "freehand" PCOC will serve as a valuable practice in the reflection and development of students' skills related to mathematical learning.

Before presenting the final considerations of this article, we thought it would be beneficial for mathematical education and the dissemination of knowledge to provide a brief pedagogical discussion of our experience with the PCOC at GPEMAC[9].

Previous experiences with the production and use of PCOC

The PCOC were created at GPEMAC through teaching, research, and extension projects developed within the group with physical and technological support from the Mathematical Visualization Laboratory (Laboratório de Visualização Matemática - L@VIM) at UESC (see L@VIM in group address). In addition to aggregating scientific initiation and course completion activities, the PCOC have been investigated and integrated into postgraduate research, such as the master's dissertations: "Prototipagem Rápida de PCOC na Impressora 3D para o Ensino e Aprendizagem de Integrais Duplas e Triplas" [Rapid Prototyping of PCOC on a 3D Printer for Teaching and Learning Double and Triple Integrals] by Marques (2016); "Aplicação de Modelos de PCOC na Aprendizagem da Geometria Espacial no Ensino Médio" [Application of PCOC Models in Learning Spatial Geometry in High School] by Ramos (2018); "O Estudo de Superfícies Regradas Mediado por Modelos De PCOC no Ensino Superior" [The Study of Ruled Surfaces Mediated by PCOC Models in Higher Education] by Sousa (2021). The experimental devices that constitute the didactic sequences of each research used as investigative instruments with students reveal the pedagogical potential of each PCOC and the formative pact that are notable in the effective practices of students who use the PCOC. Additionally, the PCOC comprise the L@VIM collection, which has received several visits from students and teachers from various basic education schools in the UESC region of influence, as well as Professors and students from the university who brought new experiences

[9] Research Group on Teaching and Learning Mathematics in a Computational Environment (Grupo de Pesquisa em Ensino e Aprendizagem da Matemática em Ambiente Computacional). www.gpemac.com.br

in teaching and learning in classes. Therefore, we hope that those actions, among others that we could detail here, such as exhibitions at local and national events and the results produced in our research, are valuable and practiced in other institutions and research groups, contributing to strengthening the debate on the integration of digital technologies in the teaching and learning of mathematics at all levels.

Final considerations

The work presented here is a frequent practice in the research we have been developing, which has contributed to the collection of the Mathematical Visualization Laboratory (L@VIM) at UESC. In effect, it enables us to answer the question we posed at the beginning: How can we support the Teacher in teaching and the student in learning GEOPAES knowings, favoring the visualization and interpretation of geometric figures presented in textbooks within the educational institution where this book is used? Therefore, the PCOC, as freely accessible teaching materials, can indeed assist teachers in teaching and students in learning mathematics, especially in GEOPAES courses in all educational institutions, as well as in Differential and Integral Calculus courses in higher education.

We therefore affirm that a 3D printed PCOC can contribute to the identification or recognition of notable geometric elements or objects in the model itself, in its manipulation and positioning related to the orthogonal Cartesian coordinate system, in the identification of the best orientation of the coordinate axes, in the application of the “tomographic” procedure, therefore examining the intersections of surface sieves that delimit it with the planes parallel to the coordinate planes, thus favoring the analysis of traces and contour lines, among other procedures or strategies that the Teacher and the students can use.

For example, the intuitive idea of curves usually treated in two-dimensional space is a notion that students typically visualize and learn more easily; however, treating a curve as being the result of the intersection of two surfaces, as occurs with an elliptic hyperboloid of one sheet and a plane, leads the student to relate the intuitive idea that they already have with the new information that this intersection imposes in three dimensions, as taught knowing (level 3) of the didactic transposition. At this point, the knowledge or knowing learned (level 4) can be realized by manipulating an elliptical hyperboloid of one sheet, such as a 3D printed PCOC. Simply reading or saying that “when examining the intersections of an elliptical one-sheeted hyperboloid with planes parallel to the coordinate planes we can see that a one-sheet hyperboloid is the meeting of hyperbolas and straight lines, or the meeting of ellipses or even the meeting of circles” does not have the same accessibility for the student at level 4 of the didactic transposition as if the Teacher carries out the same analysis in this process using the 3D printed PCOC that can be manipulated freehand.

These actions therefore prove that having a PCOC model in hand makes a significant difference in the process of acquiring knowledge based on the consideration or observation of the drawing of the corresponding geometric figure, presented in the textbook which, in many cases, appears as something given, without explaining the techniques used to obtain it, when the author of the book writes: “the figure on the side shows”.

Therefore, our reader interested in the solutions outlined in this article can help disseminate the possibilities of the production and acquisition of knowledge targeted at different educational institutions. Although the use of 3D printers in education is still a novelty, producing teaching materials and managing codes for 3D printing and being able to analyze the drawing of the corresponding geometric figure presented in the textbook or produced by the Teacher on the board is a sustainable help in the development of competencies, in the acquisition and consolidation of mathematical knowledge.

It is also worth mentioning that in the process of producing or modeling a PCOC, it is possible to see the interrelations between the mathematical knowledge inherent in the different representation registers, such as the knowings of the hyperbolic hyperboloid expressed in its general equation in the algebraic register, in its description as a mathematical object in the mother tongue, the representation of its sieves in the graphic register, the identification and representation of its traces, which are ellipses and the corresponding level curves, in the algebraic and graphic register and the mother tongue, as well as the transposition of this knowledge into a computational environment.

Indeed, algebra and geometry, as domains of basic knowings in mathematics, are always present. This mention reminds us of the speech or question of a high school student during visits organized by GPEMAC at L@VIM. After carrying out the activities proposed during one of these visits using the PCOC from the L@VIM collection, she stated, surprised by her own words: “Ah! So can we mix spatial geometry with analytic geometry?”, which reveals that many actions need to be taken to change the school reality.

References

Artigue, M. (1988). Ingénnierie didactique. Recherches en Didactique de Mathematiques. França, v. 9, no 3, p. 245-308.

Bridoux, S., Nihoul, C. (2015). *Difficultés des élèves à interpréter des constructions dans l’espace*. Une étude de cas. Petit x, nº 98.

Brousseau G. (1998). Théorie des Situations Didactiques. La Pensée sauvage, éditions, BP 141, F38100 Grenoble. ISBN 2 85919 134 8. Textes rassemblés et préparés par Nicolas Balacheff, Martin Cooper, Rosamund Sutherland et Virginia Warfield.

Camargo, I.; Boulos, P. (2005). *Geometria Analítica, um tratamento vetorial* 3a ed. Pearson.

Chevallard (Yves). (1985). *La Transposition didactique : du savoir savant au savoir enseigné*/ Yves Chevallard. Grenoble : La Pensée Sauvage, 126 p. 22 NOTES CRITIQUES cm. (Recherches en didactique des mathématiques).

Chevallard, Y. (1992). Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques,* V. 12, n°1, p. 73-112.

Duval R. (1993), Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives*. IREM de Strasbourg, v. 5, p. 35-65.

Duval R. (1995), *Sémiosis et pensée humaine*, Bern : Peter Lang.

Duval, R. (2012). Quais teorias e métodos para a pesquisa sobre o ensino da matemática? *Práxis Educativa*, Ponta Grossa, v. 7, n. 2, p. 305-330, jul./dez.

Farias, E. S; Funato, R. L; Cattai, A. P. (2016). Análise de elementos institucionais visando a prototipagem rápida na impressora 3D para o ensino e aprendizagem de integrais múltiplas. *I LADIMA*.

Henriques, Afonso; Farias, Elisângela Silva; Funato, Rosane Leite. (2024). A geometria analítica como aliada importante na aprendizagem em cálculo diferencial e integral: o caso de integrais múltiplas nos cursos de engenharias. Revista Ensino em Debate, Fortaleza, v. 2, p. e2024005. https://sites.google.com/site/gpemac/dissertacoes-de-mestrado.

Henriques, A. & Almouloud, S. A., (2022). O modelo praxeológico de gestão de tarefas imerso em um percurso de estudo e pesquisa em face de aprendizagem de Superfícies Quádricas no ensino superior envolvendo ambiente computacional. p. 361. In. Percursos de estudo e pesquisa à luz da teoria antropológica do didático: fundamentos teórico-metodológicos para a formação – volume 2 / Saddo Ag Almouloud, Renato Borges Guerra, Luiz Marcio Santos Farias, Afonso Henriques, Jose Messildo Viana Nunes (org.) Curitiba: CVR, 2022.

Henriques, A. (2021a). *Introdução ao Maple enquanto sistema de computação algébrica & gestão de códigos para impressora 3D*. Ilhéus, BA: Editus, 247 p.

Henriques, A. (2021b). Abordagem Instrumental e aplicações. *Educação Matemática Pesquisa*, São Paulo, v.23. p. 247-280.

Henriques, A. (2019). *Saberes Universitários e as suas relações na Educação Básica - Uma análise institucional em torno do Cálculo Diferencial e Integral e das Geometrias*. Via Litterarum. Ibicaraí, Bahia. Editora.

Henriques, A., Nagamine, A., Serôdio, R. (2020). Mobilização de crivos de curvas e de superfícies na resolução de problemas matemáticos: uma aplicação no ensino superior. *Educ. Matem. Pesq.*, São Paulo, v.22, n. 1, 253-275.

Henriques, A.; Nagamine, A.; Nagamine, C. M. L. (2012). Reflexões Sobre Análise Institucional: o caso do ensino e aprendizagem de integrais múltiplas. *Bolema*, Rio Claro (SP), v. 26, n. 44, p. 1261-1288, dez.

Henriques, A. (2001). *Dinâmica dos Elementos da Geometria Plana em Ambiente Computacional Cabri-Géomètre II*, Editus.

Marques, S. A. S. S. (2016) *Prototipagem rápida de PCOC na impressora 3D para o ensino e aprendizagem de integrais duplas e triplas.* [Dissertação de mestrado Acadêmico em Educação Matemática, Universidade Estadual de Santa Cruz (UESC)]: https://drive.google.com/file/d/1gxEzEhv_vCXezYXbYQGznAs39HHAFb6V/view.

Mathé. Anne-Cécile, Doze, Joris Mithalal-Le. L'usage des dessins et le rôle du langage en géométrie : quelques enjeux pour l'enseignement. La Pensée Sauvage. Nouvelles perspectives en didactique : géométrie, évaluation des apprentissages mathématiques, 1, 2019. hal-03473356

Palles, C. M.; Silva, M. J. F. (2012). Visualização em Geometria Dinâmica. *Anais do Encontro de Produção Discente*, PUC/SP/Cruzeiro do Sul. São Paulo. P. 1-9.

Parzysz, B. Articulation entre perception et déduction dans une démarche géométrique en PE1. Dans des Environnements papier – Crayon et Informatique. Extraído do Atas do 29º **Colóquio Inter-IREM des Formateurs et Professeurs charges de la Formation des maîtres**. Tours, 2002, p. 85 - 92. Ed. Univercité de Montpellier.

Ramos, M. S. (2018) *Aplicação de Modelos de PCOC na Aprendizagem da Geometria Espacial no Ensino Médio*. [Dissertação de mestrado Acadêmico em Educação Matemática, Universidade Estadual de Santa Cruz (UESC)].

https://www.biblioteca.uesc.br/pergamumweb/vinculos/201611811D.pdf.

Salazar, J. V. F.; Vita, A. C.; Almeida, T. C. S. (2008). Visualização em Geometria Espacial: Uma abordagem usando Cabri 3D. *2º Simpósio Internacional de Pesquisa em Educação Matemática - SIPEMAT*. 28 de julho a 1 de agosto.

O Estudo de Superfícies Regradas Mediado por Modelos De PCOC no Ensino Superior" de Sousa (2021). [Dissertação de mestrado Acadêmico em Educação Matemática, Universidade Estadual de Santa Cruz (UESC)]: https://drive.google.com/file/d/18RL6C_gle-Xo3H4dew59Tm_2VGbYe3XI/view